Chapter 5
Model-Based Measurement of Sector Concentration Risk in Credit Portfolios*

5.1 Fundamentals and Research Questions on Sector Concentration Risk

As demonstrated in Chap. 2, within the ASRF model it is assumed that there exists only one single risk factor that influences the defaults of all loans in the portfolio (assumption B). Thus, industry-specific or geographical effects are neglected, which can lead to an inappropriate capital requirement for real-world portfolios if this is measured on the basis of a single-factor model like the IRB Approach of Pillar 1. Against this background, banks are demanded to measure concentration risks and "explicitly consider the extent of their credit risk concentrations in their assessment of capital adequacy under Pillar 2"[267] of Basel II, but it is not specified how this should be done. Although there exist some models that explicitly deal with the measurement of sector concentration risk, these are mostly not consistent with Pillar 1 of Basel II – sometimes within the derivation and sometimes within the implementation. Consequently, it remains unclear if or how much additional regulatory capital is needed regarding risk concentrations. However, this issue is not only relevant from a regulatory perspective. Generally, it is not worthwhile to have a major gap between the regulatory and the "true" economic capital. A homogenization of these values is one goal of the new Capital Accord and would simplify the management of the credit portfolio.

In order to measure sector concentration risk consistent with the Basel II framework, it has to be reconsidered that the IRB Approach was calibrated on well-diversified bank portfolios.[268] Thus, the additional capital requirement concerning

*The main results of this section comply with Gürtler et al. (2010).

[267]BCBS (2005a), § 773.

[268]Cf. Sect. 3.3.

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sector concentrations has to take this specific calibration of the model used for calculating the Pillar 1 capital requirement into consideration. Consequently, only banks with a lower diversification across sectors than these well-diversified banks should assess additional capital under Pillar 2. As data on the characteristics of these well-diversified portfolios is not publicly available, it is not obvious how we can use them as a reference portfolio in order to modify and adjust the existing models on sector concentration risk to achieve consistency to the Basel framework. Furthermore, comparative analyses on models which are able to measure sector concentration risk are scarce. Against this background, we address the following questions:

- How can the existing approaches be modified and adjusted to be consistent with the Basel framework? Is the risk measure Value at Risk problematic when dealing with sector concentration risk?
- Which methods are capable of measuring concentration risk and how good do they perform in comparison? What are the advantages and disadvantages of these methods?

Subsequently, we propose a methodology how multi-factor models can be used in a way that is consistent with the Basel II framework. This can be seen as expanding the validity of the Basel formula from the inner region of Fig. 3.2 to the whole region. To our best knowledge, this is the first work that deals with this problem.[269] Furthermore, we analyze the models of Pykhtin (2004), Cespedes et al. (2006), and Düllmann (2006), which are designed to measure sector concentration risk. We implement our multi-factor setting for these models and use the risk measure ES instead of the VaR, which leads to some new approximation formulas. Based on this, we compare the accuracy and runtime of the different models within a simulation study. Except the rather brief analysis of Düllmann (2007), this is the first comparison of different approaches concerning sector concentration risk. In this context, we also use our framework to test whether the lack of coherency of the widespread used VaR is relevant in connection with the measurement of concentration risk.[270] Since the non-coherency of the VaR is typically only illustrated in contrived portfolio examples, we analyze the relevance of this issue in more realistic settings within our simulation study.

[269]The multi-factor model of Cespedes et al. (2006) is also specified against the background of the regulatory capital formula. However, within the deriviation of their formulas, the authors assume the regulatory capital requirement to be the upper barrier of risk, which is not consistent with the view of supervisors that we presented in Sect. 3.3 and especially in Fig. 3.2. Cf. Sect. 5.2.3 for details regarding this issue.

[270]Cf. Sect. 2.2.3.

5.2 Incorporation of Sector Concentrations Using Multi-Factor Models

5.2.1 *Structure of Multi-Factor Models and Basel II-Consistent Parameterization Through a Correlation Matching Procedure*

To obtain a more realistic modeling of correlated defaults in a credit portfolio, we will introduce a typical *multi-factor model*. In such a model, the dependence structure between obligors is not driven by one global systematic risk factor but by sector specific risk factors. Additionally, the group of obligors is divided into S sectors. Hereby, a suitable sector assignment is important,[271] i.e. asset correlations shall be high within a sector and low between different sectors. In contrast to the single-factor model, in which the correlation structure of each firm is completely described by ρ, in a multi-factor model we distinguish between an *inter-sector correlation* ρ_{Inter} and an *intra-sector correlation* ρ_{Intra}. The inter-sector correlation describes the correlation between the sector factors and the intra-sector correlation characterizes the sensitivity of the asset return to the corresponding sector factor. Thus, the asset return of obligor i in sector s can be represented by[272]

$$\tilde{a}_{s,i} = \sqrt{\rho_{\text{Intra},i}} \cdot \tilde{x}_s + \sqrt{1 - \rho_{\text{Intra},i}} \cdot \tilde{\xi}_i, \tag{5.1}$$

where $\tilde{x}_s$ is the sector risk factor (with $s = 1, \ldots, S$), and $\tilde{\xi}_i$ stands for the idiosyncratic factor. The variables $\tilde{x}_s$ and $\tilde{\xi}_i$ are normally distributed variables with mean zero and standard deviation one that are independent among each other. Since the sector risk factors $\tilde{x}_s$ are potentially dependent random variables that are difficult to deal with,[273] we make use of the possibility to present the sector risk factors as a combination of independently and standard normally distributed factors $\tilde{z}_k$ $(k = 1, \ldots, K)$

$$\tilde{x}_s = \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k \quad \text{with} \sum_{k=1}^{K} \alpha_{s,k}^2 = 1, \tag{5.2}$$

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[271]As shown by Morinaga and Shiina (2005), an assignment of borrowers to the wrong sectors usually leads to a higher estimation error than a non-optimal sector definition.

[272]In order to allow for negative intra-sector correlations, the factor loading could also be written as r_i instead of $\sqrt{\rho_{\text{Intra},i}}$. However, it is economically reasonable to assume that there is a positive relationship between the asset return of an obligor and the corresponding industry-sector. Thus, the chosen notation should be no practical limitation.

[273]Concretely, the independence of the risk factors is essential for the derivation of the Pykhtin-model in Sect. 5.2.2.

in which the factor weights $\alpha_{s,k}$ are calculated via a Cholesky decomposition of the inter-sector correlation matrix.[274] Hence, the inter-sector correlation is given as

$$\rho_{s,t}^{\text{Inter}} := \text{Corr}(\tilde{x}_s, \tilde{x}_t) = \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k}. \tag{5.3}$$

From (5.1) and (5.2), the asset correlation between obligors i in sector s and obligor j in sector t is given by

$$\text{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right) = \begin{cases} 1 & \text{if } s = t \text{ and } i = j, \\ \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} & \text{if } s = t \text{ and } i \neq j, \\ \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} \cdot \sum\limits_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} & \text{if } s \neq t. \end{cases} \tag{5.4}$$

Obligors in the same sector are highly correlated with one another when their intra-sector correlation is high. The correlation of obligors in different sectors also depends on the factor weights, which are derived from the inter-sector correlation. Hence, the dependence structure in the multi-factor model is completely described by the intra- and inter-sector correlations.

Taking (2.8) into account, the portfolio loss distribution can be written as

$$\tilde{L} = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD_{s,i} \cdot 1_{\{\tilde{a}_{s,i} < \Phi^{-1}(PD_{s,i})\}}, \tag{5.5}$$

where n_s is the number of obligors in sector s. The portfolio loss distribution can be determined numerically with Monte Carlo simulations. The procedure is in principle the same as described in Sect. 2.4 in context of the Vasicek one-factor model. In each simulation run, the sector factors as well as the idiosyncratic factor of each obligor are randomly generated. Herewith, the asset return is calculated according to (5.1). If $\tilde{a}_{s,i}$ is less than a threshold given by $\Phi^{-1}(PD_i)$, obligor i defaults. The portfolio loss is determined with (5.5) by summing up the exposure weights w_i multiplied by LGD_i of each defaulted credit. To get a good approximation of the "true" loss distribution, we choose 500,000 runs for our subsequent Monte Carlo simulations. After running the simulation and sorting the loss outcomes, we get the portfolio loss distribution. The ES at a given confidence level α can be calculated with (2.47).

To calibrate the multi-factor model, most variables can be chosen identically to the single-factor model. The only difference is the correlation structure that

[274] This approach is a common mathematical method to generate correlated random variables and leads to the identical number of independent risk factors $\tilde{z}_k$ and dependent sector factors $\tilde{x}_s$, that is K equals S. Another common method to determine independent risk factors is the principal component analysis, which leads to a reduced number of risk factors.

generally consists of inter- and intra-sector correlations as described above. The matrix of inter-sector correlations is usually derived from historical default rates or from equity correlations between industry sectors. The intra-sector correlations can be derived from historical default rates, too. The problem of a derivation based on historical default rates is that there are not always enough observations to get stable results. This is even more problematic if it is assumed (like in Basel II) that the correlation and the PD are interdependent. Furthermore, the results from the multi-factor model would normally not be consistent with Basel II because the correlation structure is completely different. Thus, it would not be possible to identify (consistent with Pillar 1 of Basel II) whether there is need for additional regulatory capital under Pillar 2.

For both reasons, the intra-sector correlations could be chosen analogously to the Basel II formula

$$\rho_{\text{Basel}} = 0.12 \cdot \frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}} + 0.24 \cdot \left(1 - \frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}}\right) \tag{5.6}$$

for corporates. This is what Cespedes et al. (2006) did in their analyses. But this assumption is critical for the following reason: The validity of this formula for the intra-sector correlations is equivalent to the statement that the regulatory capital calculated via the formula of Pillar 1 is an upper barrier of the true risk. This property in turn is only fulfilled if either only one sector exists or if all sectors are perfectly correlated. In all other cases there is an effect of sector diversification, which leads to a lower capital requirement compared to the Basel framework. Beyond, the Basel Committee does not intend the Basel II correlation formula to exclusively reflect the intra-sector correlation. Instead, the framework is calibrated on well-diversified portfolios, as demonstrated in Fig. 3.2, implying that the correlation formula is chosen in a way that the single-factor model leads to a good approximation of the "true" risk based on the full correlation structure in a multi-factor model. Cespedes et al. (2006) have already recognized this criticism and have mentioned that it should be possible to use some scaling up for the intra-sector correlations and the resulting capital. However, their calculations are based on the formula above.

Alternatively, the intra-sector correlation could be chosen in a way that the regulatory capital RC can be matched with the economic capital EC^{mf}, which is simulated for a well-diversified portfolio within a multi-factor model. Therefore, we define the "*implicit intra-sector correlation*" $\rho_{\text{Intra}}^{(\text{Implied})}$ by

$$EC^{\text{mf}}\left(\rho_{\text{Inter}}, \rho_{\text{Intra}}^{(\text{Implied})}\right) = RC(\rho_{\text{Basel}}). \tag{5.7}$$

Unfortunately, the portfolios for which the calibration was done by the Basel Committee including the assumed inter-sector correlation structure are not publicly available. Thus, at first we have to choose a concrete inter-sector correlation and determine the implicit intra-sector correlation for some hypothetical, well-diversified

Table 5.1 Inter-sector correlation structure based on MSCI industry indices (in %)[a]

Sector	A	B	C1	C2	C3	D	E	F	H	I	J
A: Energy	100	50	42	34	45	46	57	34	10	31	69
B: Materials		100	87	61	75	84	62	30	56	73	66
C1: Capital goods			100	67	83	92	65	32	69	82	66
C2: Comm. svs. and supplies				100	58	68	40	8	50	60	37
C3: Transportation					100	83	68	27	58	77	67
D: Consumer discretionary						100	76	21	69	81	66
E: Consumer staples							100	33	46	56	66
F: Health care								100	15	24	46
H: Information technology									100	75	42
I: Telecommunication services										100	62
J: Utilities											100

[a]See Düllmann and Masschelein (2007), p. 64

portfolios via Monte Carlo simulations with several parameter trials. This approach is related to Lopez (2004), who empirically determines the single correlation parameter for the ASRF model that leads to the same 99.9%-quantile as KMV's multi-factor model for several portfolio types (geographical region, PD, and asset size categories) using a grid search procedure. Thus, in the approach of Lopez (2004), the left-hand side of (5.7) is given and the single correlation parameter of the right-hand side is determined, whereas we are searching for the intra-sector correlation on the left-hand side that leads to a match of both models when the other parameters, especially the single correlation parameter of Basel II, are exogenously given.

As mentioned above, the required inter-sector correlation matrix could be estimated from historical default rates or from time series of stock returns.[275] Düllmann et al. (2008) demonstrate on the basis of an extensive simulation study that it is recommendable to use stock prices instead of historical default rates since this involves smaller statistical errors. Against this background, we rely on equity correlations, too, and use the correlation matrix of the MSCI EMU industry indices computed by Düllmann and Masschelein (2007) for the inter-sector correlation structure (see Table 5.1).[276]

Our definition of a well-diversified portfolio is based on the overall sector concentration of the German banking system, which can be found in Table 5.2.[277]

Even if it is theoretically possible to achieve lower capital requirements through a different sector decomposition, this can only be done by a restricted number of banks, since a deviation from the market structure of all banks immediately leads to a disequilibrium. In addition, the total number of credits is assumed to be $n = 5{,}000$ to guarantee low granularity.

[275]An overview of the literature regarding the measurement of asset correlation parameters can be found in Düllmann et al. (2008) and Grundke (2008).

[276]The correlation structure based on the MSCI US is similar, see Düllmann and Masschelein (2007).

[277]Düllmann and Masschelein (2007) notice that the concentration is very similar to other countries like France, Belgium, and Spain.

Table 5.2 Overall sector composition of the German banking system[a]

Sector	Exposure weight (%)
A: Energy	0.18
B: Materials	6.01
C1: Capital goods	11.53
C2: Comm. svs. and supplies	33.69
C3: Transportation	7.14
D: Consumer discretionary	14.97
E: Consumer staples	6.48
F: Health care	9.09
H: Information technology	3.20
I: Telecommunication services	1.04
J: Utilities	6.67

[a]Cf. Düllmann and Masschelein (2007), p. 63

Table 5.3 Implicit intra-sector correlations for different portfolio qualities

Portfolio type/quality	Implicit intra-sector correlation (%)
(I) Very high	30
(II) High	28
(III) Average	25
(IV) Low	23
(V) Very low	21

If we assume a constant intra-sector correlation, the best match is achieved by (approximately) $\rho_{\text{Intra}}^{(\text{Implied})} = 25\%$.[278] The concrete results, however, vary with the portfolio quality (see Table 5.3).[279] Thus, using a constant intra-sector correlation can lead to a significant underestimation of economic capital for high-quality portfolios and to an overestimation for low-quality portfolios.

To reduce the deviation, the intra-sector correlation should be decreasing in PD. We found that the following intra-sector correlation function leads to a good match for portfolios with different quality distributions:

$$\rho_{\text{Intra}}^{(\text{Implied})} = 0.185 \cdot \frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}} + 0.34 \cdot \left(1 - \frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}}\right). \tag{5.8}$$

Thus, we use the correlation function type from Basel II but the correlation range is from 18.5 to 34% instead of 12 to 24%.[280] It has to be noted that this formula is

[278]This value results on the basis of both measures (VaR and ES) at the respective confidence level as described in Sect. 4.3.1. The result is consistent with Düllmann and Masschelein (2007), who use a constant intra-sector correlation of 25% in their analysis.

[279]See Fig. 4.7 for the portfolio characteristics.

[280]We tried several different functional forms but the formula above performed best. The multipliers 18.5% and 34% in function (5.8) were determined with a grid search using a reasonable parameter range, which is similar to the procedure of Lopez (2004) used for the single correlation parameter.

still a substantial simplification, as we assume that the intra-sector correlation is PD-dependent only. By contrast, empirically there are also inter-sectoral differences of this parameter.[281] In principle it would be possible to capture both effects, e.g. by multiplying a sector-specific factor to (5.8), which covers the relation of the empirically observed correlations.[282] Of course, the absolute level of the resulting correlations would usually be different from the empirical observations to keep Basel II consistent results. But for convenience, we rely on the PD-dependent formula (5.8) in our following analyses.

Hence, all additional input data needed for typical multi-factor models, e.g. using Monte Carlo simulations, are given with Table 5.1 and (5.8). Using these values, the multi-factor models should be consistent with the Basel framework. Thus, the measured economic capital is only lower than the regulatory capital if the portfolio is less concentrated than a typical, well-diversified portfolio, and the needed economic capital is above the capital requirement of the regulatory framework if there is more concentration risk in the credit portfolio. In order to avoid time-consuming Monte Carlo simulations, there exist some multi-factor models for an approximation of the portfolio risk. These will be presented subsequently.

5.2.2 *Accounting for Sector Concentrations with the Model of Pykhtin*

5.2.2.1 Derivation of the VaR-Based Multi-Factor Adjustment

In this section, the multi-factor adjustment of Pykhtin (2004) is examined. After explanation of the approach and derivation of the multi-factor adjustment formula for the VaR, the ES-based formula is calculated. Since the main shortcoming of the model is the time-consuming calculation for large portfolios, we focus on this issue thereafter and demonstrate how the approach can be implemented in a way that calculation time is reduced significantly.[283]

The multi-factor adjustment is an extension of the granularity adjustment presented in Chap. 4, which was introduced by Gordy (2003), Wilde (2001), and Martin and Wilde (2002), for multi-factor models and provides an analytical method for calculating the VaR and ES of a credit portfolio. The basic idea of Pykhtin is to approximate the portfolio loss $\tilde{L}$ in the multi-factor model with the respective portfolio loss $\tilde{\bar{L}}$ in an accurately adjusted ASRF model. This is done by

[281] E.g. Heitfield et al. (2006) determine the sector loadings, which equal $\sqrt{\rho_{\text{Intra}}}$, for 50 industry sectors using KMV data on asset values. The resulting intra-sector correlation is on average 18.8% and the standard deviation is 8.3%. These inter-sectoral differences are not captured by the formula above.

[282] A correlation structure with one degree of freedom for every PD/sector-combination is practically unfeasible due to high data requirements.

[283] In our setting, the computation time could be reduced by more than 99.8%.

mapping the correlation structure of all credits in the multi-factor model into a single correlation factor. This factor is determined by maximizing the correlation between the new single risk factor $\tilde{\tilde{x}}$ and the original sector factors $\{\tilde{x}_s\}$. Based on this, a Taylor series expansion is performed around the constructed single-factor model.

Concretely, the distribution of $\tilde{\tilde{L}}$, which is the loss of the accurately adjusted single-factor model, can be calculated with the known formula of the ASRF model:[284]

$$\tilde{\tilde{L}} = \mu_{1,c}\left(\tilde{\tilde{x}}\right) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \Phi\left[\frac{\Phi^{-1}(PD_i) - c_i \cdot \tilde{\tilde{x}}}{\sqrt{1 - c_i{}^2}}\right], \tag{5.9}$$

where c_i is the correlation between the asset returns of two obligors, which is due to the conjoint dependence to the systematic risk factor $\tilde{\tilde{x}}$. Instead of using ρ as an input parameter as it is done in the ASRF model, the new correlation parameter c_i is calculated in a way that the correlation between the introduced single risk factor $\tilde{\tilde{x}}$ and the original sector factors $\{\tilde{x}_s\}$ is maximized. Thus, most of the correlation structure in the multi-factor model should be matched by this single factor.

As a next step, a Taylor series expansion around the comparable one-factor model (5.9) is performed in order to reduce the approximation error. Via this approach, it is possible to approximate the α-quantile $q_\alpha\left(\tilde{L}\right)$ of the portfolio loss by

$$q_\alpha\left(\tilde{L}\right) \approx q_\alpha\left(\tilde{\tilde{L}}\right) + \lambda \cdot \left[\frac{dq_\alpha\left(\tilde{\tilde{L}} + \lambda\tilde{Z}\right)}{d\lambda}\right]_{\lambda=0} + \frac{\lambda^2}{2} \cdot \left[\frac{d^2 q_\alpha\left(\tilde{\tilde{L}} + \lambda\tilde{Z}\right)}{d\lambda^2}\right]_{\lambda=0}, \tag{5.10}$$

where λ is the scale of perturbation and $\lambda\tilde{Z}$ describes the approximation error between "true" loss $\tilde{L}$ and the loss in the comparable one-factor model $\tilde{\tilde{L}}$, i.e. $\tilde{L} - \tilde{\tilde{L}} =: \lambda\tilde{Z}$. The first summand on the right-hand side of (5.10) is the α-quantile of the loss $\tilde{\tilde{L}}$ within the reasonably adjusted ASRF model, which is $\mu_{1,c}(\Phi^{-1}(1 - \alpha))$.[285] The required correlation factor c_i is derived in Appendix 5.5.1.[286] In addition to maximizing the correlation between the single factor and the sector factors, the concrete choice of c_i guarantees that the first derivative in (5.10) is equal to zero, see also Appendix 5.5.1. Hence, the so-called multi-factor adjustment Δq_α is completely described by the second derivative in (5.10). According to Pykhtin (2004), the *multi-factor adjustment* Δq_α can be written as[287]

[284]The conditional PD stems from the Vasicek model, cf. Sect. 2.4 or 2.7.

[285]Cf. (5.9).

[286]For the determination of c_i, we need both the intra- and inter-sector correlations, which can be taken from Sect. 5.2.1.

[287]This formula has already been derived for the granularity adjustment formula, cf. (4.18).

$$\begin{aligned}\Delta q_\alpha &= q_\alpha(\tilde{L}) - q_\alpha\left(\tilde{\tilde{L}}\right) \\ &\approx -\frac{1}{2\cdot d\mu_{1,c}(\overline{x})/d\overline{x}}\cdot\left[\frac{d\eta_{2,c}(\overline{x})}{d\overline{x}} - \eta_{2,c}(\overline{x})\cdot\left(\frac{d^2\mu_{1,c}(\overline{x})/d\overline{x}^2}{d\mu_{1,c}(\overline{x})/d\overline{x}} + \overline{x}\right)\right]\Bigg|_{\overline{x}=\Phi^{-1}(1-\alpha)},\end{aligned} \tag{5.11}$$

where $\eta_{m,c}(\overline{x}) := \eta_m(\tilde{L}|\tilde{\overline{x}} = \overline{x})$ is the mth conditional moment of the portfolio loss about the mean.

The conditional expectation $\mu_{1,c}(\overline{x})$ and the required derivatives are already known from the granularity adjustment:[288]

$$\mu_{1,c}(\overline{x}) = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot p_i(\overline{x}), \tag{5.12}$$

$$\frac{d\mu_{1,c}(\overline{x})}{d\overline{x}} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{d(p_i(\overline{x}))}{d\overline{x}}, \tag{5.13}$$

$$\frac{d^2\mu_{1,c}(\overline{x})}{d\overline{x}^2} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{d^2(p_i(\overline{x}))}{d\overline{x}^2}, \tag{5.14}$$

with

$$p_i(\overline{x}) = \Phi\left(\frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-c_i{}^2}}\right), \tag{5.15}$$

$$\frac{d(p_i(\overline{x}))}{d\overline{x}} = -\frac{c_i}{\sqrt{1-c_i{}^2}}\cdot\varphi\left(\frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-c_i{}^2}}\right), \tag{5.16}$$

and

$$\frac{d^2(p_i(\overline{x}))}{d\overline{x}^2} = -\frac{c_i{}^2}{1-c_i{}^2}\cdot\frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-c_i{}^2}}\cdot\varphi\left(\frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-c_i{}^2}}\right). \tag{(5.17)}$$

The conditional variance $\eta_{2,c}$ is

$$\eta_{2,c} = \mathbb{V}\left(\sum_{i=1}^{n} w_i\cdot\widetilde{LGD}_i\cdot 1_{\{\tilde{D}_i\}}|\tilde{\overline{x}} = \overline{x}\right) \tag{5.18}$$

[288] Cf. Sect. 4.2.1.2.

but in contrast to the single risk-factor framework, the defaults are not independent conditional on $\bar{x}$. Thus, it is not possible to use the formula of the granularity adjustment. The dependence structure of the conditional default events becomes apparent if we rewrite the formula of the asset return (5.1) using (5.2) and (5.73):

$$\begin{aligned}
\tilde{a}_{s,i} &= \sqrt{\rho_{\text{Intra},i}} \cdot \tilde{x}_s + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\
&= \sqrt{\rho_{\text{Intra},i}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\
&= c_i \cdot \tilde{\bar{x}} - c_i \cdot \tilde{\bar{x}} + \sqrt{\rho_{\text{Intra},i}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\
&= c_i \cdot \tilde{\bar{x}} - c_i \cdot \sum_{k=1}^{K} b_k \cdot \tilde{z}_k + \sqrt{\rho_{\text{Intra},i}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\
&= c_i \cdot \tilde{\bar{x}} + \sum_{k=1}^{K} \left(\sqrt{\rho_{\text{Intra},i}} \cdot \alpha_{s,k} - c_i \cdot b_k \right) \cdot \tilde{z}_k + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i .
\end{aligned} \tag{5.19}$$

Even if the systematic factor $\tilde{\bar{x}}$ is fixed, the asset returns are not independent of each other but depend on the constructed sector variables $\tilde{z}_k$.[289] The correlation between obligor i and j conditional on $\tilde{\bar{x}}$ can be calculated as:[290]

$$\begin{aligned}
\rho_{ij}^{\bar{x}} &= \operatorname{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j} | \tilde{\bar{x}}\right) \\
&= \frac{\sqrt{\rho_{\text{Intra},i} \cdot \rho_{\text{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} - c_i \cdot c_j}{\sqrt{\left(1-c_i{}^2\right) \cdot \left(1-c_j{}^2\right)}} .
\end{aligned} \tag{5.20}$$

Although the asset returns are not independent conditional on $\tilde{\bar{x}}$, they are independent conditional on the sector factors $\tilde{z}_k$. We can use this property by decomposing the conditional variance of the portfolio loss $\eta_{2,c}(\bar{x})$ into two terms, $\eta_{2,c}^{\infty}(\bar{x})$ and $\eta_{2,c}^{\text{GA}}(\bar{x})$:[291]

$$\eta_{2,c}(\bar{x}) = \mathbb{V}\left(\tilde{L} | \tilde{\bar{x}} = \bar{x}\right) = \underbrace{\mathbb{V}\left[\mathbb{E}\left(\tilde{L} | \{\tilde{z}_k\}\right) | \tilde{\bar{x}} = \bar{x}\right]}_{\eta_{2,c}^{\infty}(\bar{x})} + \underbrace{\mathbb{E}\left[\mathbb{V}\left(\tilde{L} | \{\tilde{z}_k\}\right) | \tilde{\bar{x}} = \bar{x}\right]}_{\eta_{2,c}^{\text{GA}}(\bar{x})} . \tag{5.21}$$

The term $\eta_{2,c}^{\infty}(\bar{x})$ describes the systematic risk adjustment, which is given by the difference between the multi-factor and single-factor loss distribution in infinitely

[289] Cf. (5.2).

[290] See Appendix 5.5.2.

[291] The derivation of the variance decomposition can be found in Weiss (2005), p. 385 f.

fine-grained portfolios. The other term $\eta_{2,c}^{\mathrm{GA}}(\bar{x})$ is relevant for the granularity adjustment, which measures the influence of portfolio name concentration. The calculation of the terms $\eta_{2,c}^{\infty}(\bar{x})$ and $\eta_{2,c}^{\mathrm{GA}}(\bar{x})$ can be found in Appendix 5.5.3 and utilizes the conditional independence property of the decomposed terms. This leads to

$$\eta_{2,c}^{\infty}(\bar{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j \left[\Phi_2\left(\Phi^{-1}(p_i(\bar{x})), \Phi^{-1}\left(p_j(\bar{x})\right), \rho_{ij}^{\bar{x}}\right) - p_i(\bar{x}) p_j(\bar{x})\right], \tag{5.22}$$

$$\eta_{2,c}^{\mathrm{GA}}(\bar{x}) = \sum_{i=1}^{n} w_i{}^2 \left(ELGD_i{}^2 \left[p_i(\bar{x}) - \Phi_2\left(\Phi^{-1}(p_i(\bar{x})), \Phi^{-1}(p_i(\bar{x})), \rho_{ii}^{\bar{x}}\right)\right] + VLGD_i p_i(\bar{x})\right). \tag{5.23}$$

According to (5.11), we also need the derivative $d\eta_{2,c}(\bar{x})/d\bar{x}$. Thus, the derivatives of the decomposed variance terms are calculated in Appendix 5.5.4, leading to

$$\begin{aligned} \frac{d\eta_{2,c}^{\infty}(\bar{x})}{d\bar{x}} &= 2 \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j \frac{dp_i(\bar{x})}{d\bar{x}} \\ &\quad \cdot \left(\Phi\left(\frac{\Phi^{-1}\left[p_j(\bar{x})\right] - \rho_{ij}^{\bar{x}}\Phi^{-1}[p_i(\bar{x})]}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right) - p_j(\bar{x})\right), \end{aligned} \tag{5.24}$$

$$\begin{aligned} \frac{d\eta_{2,c}^{\mathrm{GA}}(\bar{x})}{d\bar{x}} &= \sum_{i=1}^{n} w_i^2 \frac{dp_i(\bar{x})}{d\bar{x}} \cdot \left(ELGD_i^2\left[1 - 2\Phi\left(\frac{\Phi^{-1}[p_i(\bar{x})] - \rho_{ii}^{\bar{x}}\Phi^{-1}[p_i(\bar{x})]}{\sqrt{1-\left(\rho_{ii}^{\bar{x}}\right)^2}}\right)\right]\right. \\ &\quad \left. + VLGD_i\right). \end{aligned} \tag{5.25}$$

Using the terms (5.13)–(5.17), (5.20), and (5.22)–(5.25), the multi-factor adjustment (5.11) can be calculated. Since the multi-factor adjustment is linear in the conditional variance and its derivatives, we can also write the multi-factor adjustment as

$$\Delta q_\alpha = \Delta q_\alpha^{\infty} + \Delta q_\alpha^{\mathrm{GA}}, \tag{5.26}$$

i.e. the multi-factor adjustment can be split into a systematic risk adjustment component and a granularity adjustment component. To sum up, the approximation of a loss quantile $q_\alpha(\tilde{L})$ in (5.10) is given by (5.9) and by the multi-factor adjustment

$$q_\alpha(\tilde{L}) \approx q_\alpha\left(\bar{\tilde{L}}\right) + \Delta q_\alpha = q_\alpha\left(\bar{\tilde{L}}\right) + \Delta q_\alpha^{\infty} + \Delta q_\alpha^{\mathrm{GA}}. \tag{5.27}$$

5.2.2.2 Derivation and Implementation of the ES-Based Multi-Factor Adjustment

After dealing with the VaR, now the ES-based multi-factor adjustment is presented. Using the integral representation of the ES (2.20) and substituting the quantile $q_\alpha(\tilde{L})$ by approximation (5.27), the ES can be written as

$$\begin{aligned} ES_\alpha\left(\tilde{L}\right) &= \frac{1}{1-\alpha} \cdot \int_\alpha^1 q_s(\tilde{L})ds \\ &\approx \frac{1}{1-\alpha} \cdot \int_\alpha^1 \left(q_s\left(\tilde{\bar{L}}\right) + \Delta q_s\right)ds \\ &= ES_\alpha\left(\tilde{\bar{L}}\right) + \frac{1}{1-\alpha} \cdot \int_\alpha^1 \Delta q_s ds =: ES_\alpha\left(\tilde{\bar{L}}\right) + \Delta ES_\alpha. \end{aligned} \tag{5.28}$$

The first summand of the right-hand side describes the ES for the comparable single-factor model and the second summand is the multi-factor adjustment.

The ES in the ASRF model is already known from (4.59), leading to

$$ES_\alpha\left(\tilde{\bar{L}}\right) = \frac{1}{1-\alpha} \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \Phi_2\left(-\Phi^{-1}(\alpha), \Phi^{-1}(PD_i), c_i\right). \tag{5.29}$$

In order to calculate the multi-factor adjustment in (5.28), we use the formulation of Δq_s from (4.18):

$$\Delta ES_\alpha\left(\tilde{L}\right) = -\frac{1}{2(1-\alpha)} \int_\alpha^1 \frac{1}{\varphi(\bar{x})} \frac{d}{d\bar{x}} \left(\frac{\varphi(\bar{x})\,\eta_{2,c}(\bar{x})}{d\mu_{1,c}(\bar{x})/d\bar{x}}\right)\Bigg|_{\bar{x}=\Phi^{-1}(1-s)} ds. \tag{5.30}$$

Substituting $x := \Phi^{-1}(1-s)$ and thus $ds = -\varphi(x)dx$, $x(s=\alpha) = \Phi^{-1}(1-\alpha)$, and $x(s=1) = -\infty$ results in

$$\begin{aligned} \Delta ES_\alpha\left(\tilde{L}\right) &= -\frac{1}{2(1-\alpha)} \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \frac{1}{\varphi(x)} \frac{d}{d\bar{x}} \left(\frac{\varphi(\bar{x})\,\eta_{2,c}(\bar{x})}{d\mu_{1,c}(\bar{x})/d\bar{x}}\right)\Bigg|_{\bar{x}=x} \varphi(x)dx \\ &= -\frac{1}{2(1-\alpha)} \left[\left(\frac{\varphi(x)\,\eta_{2,c}(x)}{d\mu_{1,c}(x)/dx}\right)\right]_{-\infty}^{\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{5.31}$$

The derivative of $\mu_{1,c}$ can be written as $d\mu_{1,c}(x)/dx = g \cdot \varphi(x)$, with g being a constant value. As $\eta_{2,c}(-\infty) = 0$, the right-hand side of (5.31) vanishes at $x = -\infty$, leading to

$$\Delta ES_\alpha(\tilde{L}) = -\frac{1}{2(1-\alpha)} \frac{\varphi(x)\,\eta_{2,c}(x)}{d\mu_{1,c}(x)/dx}\bigg|_{x=\Phi^{-1}(1-\alpha)}. \quad (5.32)$$

This equation can easily be computed using the conditional variance and the derivative of the conditional expectation of Sect. 5.2.2.1. Again, the multi-factor adjustment can be decomposed into a systematic and an idiosyncratic part by decomposing the conditional variance. Hence, the ES for a portfolio in a multi-factor model is given by

$$ES_\alpha(\tilde{L}) = ES_\alpha\left(\tilde{\bar{L}}\right) + \Delta ES_\alpha^\infty + \Delta ES_\alpha^{\mathrm{GA}}. \quad (5.33)$$

It is worth noticing that the resulting expression (5.32) is much simpler than the corresponding formula for the VaR. The same phenomenon could already be observed for the granularity adjustment formula in Chap. 4.

In principle, it is straightforward to implement the Pykhtin model. For calculating the ES we have to compute (5.32). The problem is that the computation can be extremely time-consuming if the formula is applied to large portfolios. The reason is that the calculation procedure inter alia requires n^2-times the computation of the conditional asset correlation,[292] with n being the number of credits. An alternative performed by Düllmann and Masschelein (2007) is to neglect the multi-factor adjustment and to use (5.9) only to aggregate all credits for each sector and thus using the formulas on sector and not on borrower level. Of course, it may be expected that this simplification is at the cost of lower approximation accuracy. To consider the multi-factor adjustment, we propose to build PD-classes for each of the sectors and aggregate the credits to these buckets for the calculation of the multi-factor adjustment, so that the computation time is predominated by

$$\mathrm{Loops} = (N_{\mathrm{PD}} \cdot S)^2, \quad (5.34)$$

where N_{PD} and S denote the number of PD-classes and sectors, respectively.[293] If the number of PD-classes is sufficient, the approximation error resulting from aggregating individual PDs to PD-classes is negligible. As the number of loops does not grow with bigger portfolios, it is possible to perform the adjustment on

[292] The quadratic computation effort is due to the determination of a double sum (see (5.22) and (5.24)).

[293] The results of the multi-factor adjustment do not differ whether different exposures with the same PD are aggregated or handled separately on borrower level. For details see Sect. 5.2.2.1 and Appendix 5.5.1.

bucket level within reasonable time. Only the granularity adjustment should be calculated on borrower level but this is no computational burden.[294]

5.2.3 Accounting for Sector Concentrations with the Model of Cespedes, Herrero, Kreinin and Rosen

5.2.3.1 Design of the Diversification Factor

Cespedes et al. (2006) present a method to relate the economic capital in the multi-factor model to the regulatory capital formula.[295] These models are linked via a *diversification factor* $DF(\cdot)$, which depends on two parameters:

- The average sector concentration HHI and
- The average weighted inter-sector correlation $\bar{\beta}$

Herewith, the economic capital of a portfolio can be approximated as:

$$EC^{\mathrm{mf}} \approx DF\left(HHI, \overline{\beta}\right) \cdot RC. \tag{5.35}$$

Thus, the economic capital in the multi-factor model EC^{mf} can be approximated by a well-defined diversification factor DF multiplied with the regulatory capital requirement of the ASRF model RC. As mentioned before, Cespedes et al. (2006) assume in their calculations the regulatory capital of Pillar 1 to be an upper barrier of the true risk because no diversification effects between the sectors are considered, which in turn implies the parameter DF to be always less than or equal to one. In contrast, if we use our definition of the intra-sector correlation ρ_{Intra} from Sect. 5.2.1, it is possible to obtain $EC^{\mathrm{mf}} > RC$ as well as $EC^{\mathrm{mf}} < RC$ depending on the degree of diversification in comparison to the well-diversified portfolio defined in Sect. 5.2.1. Hence, our later on calculated DF-function can be greater than one, i.e. the DF-function measures not only the benefit from sector diversification but also the risk resulting from high sector concentration. As the regulatory capital is additive in the ASRF model, (5.35) can be substituted by

$$EC^{\mathrm{mf}} \approx DF \cdot \sum_{s=1}^{S} RC^{s}, \tag{5.36}$$

in which EC^{mf} is the economic capital in the multi-factor model and RC^{s} is the regulatory capital for sector s. In principle, the approach can be characterized as

[294]The computation time when calculating the multi-factor adjustment on bucket- instead on borrower-level can be reduced from 67 min to 5 s for a portfolio with 11 sectors, 7 PD-classes, and 5,000 creditors.

[295]In the strict sense, Cespedes et al. (2006) relate the multi-factor model to the *economic* capital in a single-factor model. But since they apply the regulatory capital formula and we require a relation to this formula, too, we use the term regulatory capital instead.

follows: Firstly, EC^{mf} is calculated for a multitude of portfolios via Monte Carlo simulations. For each simulated portfolio, the diversification factor can be calculated according to (5.36). Finally, a regression is performed to get an approximation for DF as a function of the two parameters HHI and $\overline{\beta}$. If DF can capture the industry diversification effects, we are able to approximate EC^{mf} with (5.36) without additional Monte Carlo simulations.

To derive the parameters which explain the effect of diversification and concentration in a multi-factor model, Cespedes et al. (2006) suggest to use the average inter-sector correlation $\overline{\beta}$. This can be interpreted as a scale of the dependence between the sectors. The formula for $\overline{\beta}$ is given as

$$\overline{\beta} = \frac{\sum_{s=1}^{S} \sum_{t \neq s} \rho_{s,t}^{\mathrm{Inter}} \cdot RC^s \cdot RC^t}{\sum_{s=1}^{S} \sum_{t \neq s} RC^s \cdot RC^t}, \tag{5.37}$$

The correlation is weighted by the regulatory capital in order to account for the contribution of each sector. As a consequence, the correlation between sectors with a high capital requirement account for a high degree of the average correlation.[296] The second suggested parameter is a parameter for the degree of capital diversification, measured by the Herfindahl–Hirschmann Index HHI.[297] It describes the sector concentration measured by the relative weight of each sectors regulatory capital RC^s:[298]

$$HHI = \frac{\sum_{s=1}^{S} (RC^s)^2}{\left(\sum_{s=1}^{S} RC^s\right)^2}. \tag{5.38}$$

As mentioned in Sect. 3.4, the parameter HHI lies between two extreme values:

- $HHI = 1/S$, i.e. perfect sector diversification,
- $HHI = 1$, i.e. perfect sector concentration.

To avoid a too complex model, Cespedes et al. (2006) neglect further potential input parameters to determine the DF-function. To approximate the multi-factor model, (5.36) can be rewritten as

$$EC^{\mathrm{mf}} \approx DF\left(HHI, \overline{\beta}\right) \cdot \sum_{s=1}^{S} RC^s. \tag{5.39}$$

[296] The idea is related to Pykhtin (2004), who uses the VaR from the ASRF model as a weight when maximizing the correlation between the single factor of the comparable one-factor model and the sector factors; cf. (5.82)–(5.85).

[297] Cespedes et al. (2006) call this parameter the capital diversification index (CDI).

[298] This concentration measure corresponds to (2.87).

5.2.3.2 Computation of the Diversification Factor by Simulation

In the following, our procedure to estimate the *DF*-function is presented. In order to get a universally valid *DF*-factor, as many portfolios as possible have to be generated and simulated. To reduce the necessary number of trials, the portfolios should be restricted to those with reasonable characteristics. Our portfolios are randomly generated using the following parameter setting. When we state several parameter values or a parameter range, the parameter is randomly drawn from this set.

For the intra-sector correlations, we use the functional form of (5.8). The inter-sector correlation structure is taken from Table 5.1, so that all simulated portfolios are stemming from this sector definition. Each portfolio consists of {2, …, 11} sectors that are randomly drawn from the different industries. The sector weights are in [0, 1] and sum up to one. The total number of credits is 5,000, equally divided for each sector. Each sector in turn consists of credits from the PD classes {AAA, AA, A, BBB, BB, B, CCC}. Instead of using equally distributed PD classes, we draw the quality distribution from our predefined credit portfolio qualities {very high, high, average, low, very low} for every sector from Fig. 4.7.[299] We draw 25,000 or 50,000 portfolios and compute the economic capital in the multi-factor model for each portfolio.

To determine the economic capital, we have tried both Monte Carlo simulations with 100,000 trials[300] for every portfolio and the Pykhtin formula from Sect. 5.2.2. Because the computation time for Monte Carlo simulations is materially longer, the corresponding results are based on 25,000 random portfolios, whereas we computed the economic capital for 50,000 portfolios when using the Pykhtin formula instead. Furthermore, since Cespedes et al. (2006) use the VaR as the relevant risk measure and thus define the economic capital as $EC^{\mathrm{mf}} := VaR^{\mathrm{mf}}_{0.999} - EL$, we have to redefine the economic capital of the multi-factor model with respect to ES as argued in Sect. 4.3.1: $EC^{\mathrm{mf}} := ES^{\mathrm{mf}}_{0.9972} - EL$.[301] In contrast, for the regulatory capital we use $RC = VaR^{(\mathrm{Basel})} - EL$. The result could also be related to the Expected Shortfall in the ASRF model but we have detected that the results differ only marginally and the VaR is easier to implement in typical spreadsheet applications.[302] The results for the diversification factor *DF* are very similar regardless of whether they are based on

[299]The setting is similar to Cespedes et al. (2006). Until this point, the main difference is the definition of the intra- and inter-sector correlations.

[300]For the determination of the economic capital for one specific portfolio, the number of trials is slightly low but as we perform 25,000 simulations and the simulation noise of each simulation is unsystematic, the error terms should cancel out each other to a large extent.

[301]We have also tested the results when using the ES instead of the unexpected loss but the coefficient of determination is higher when subtracting the EL in the corresponding formulas when performing the simulations.

[302]To determine the Expected Shortfall with (4.59), a bivariate cumulative normal distribution has to be computed whereas the Value at Risk only makes use of univariate distributions.

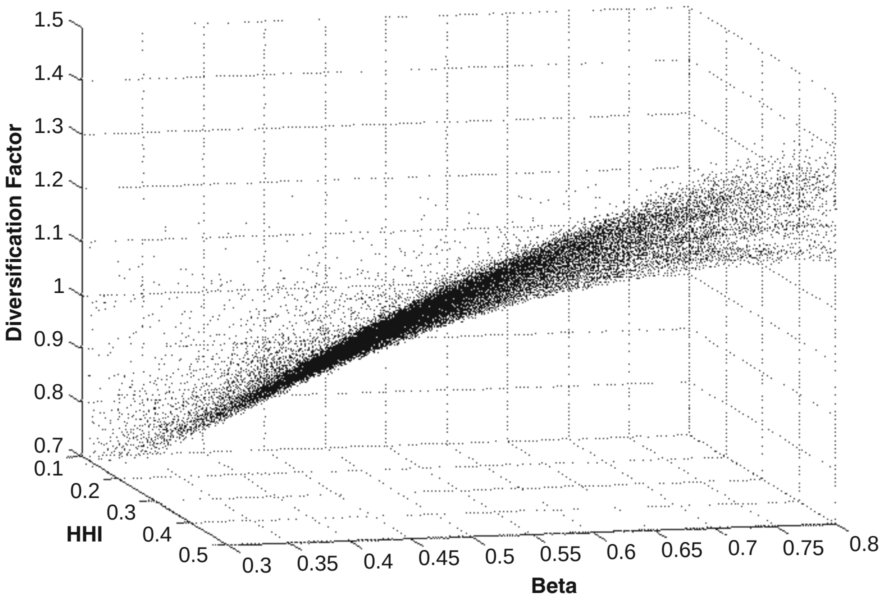

Fig. 5.1 Diversification Factor realizations on the basis of 50,000 simulations

Monte Carlo simulations or on the Pykhtin formula. Fig. 5.1 presents characteristics of the diversification factor when using the Pykhtin formula.

For a determination of the functional form of DF, we use a regression of the type[303]

$$\begin{aligned} DF = a_0 + a_1 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right) \\ + a_2 \cdot (1 - HHI)^2 \cdot \left(1 - \overline{\beta}\right) + a_3 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right)^2 \end{aligned} \tag{5.40}$$

in both cases, using the ordinary least squares (OLS) technique. The resulting function when using Monte Carlo simulations is

$$\begin{aligned} DF_{\text{MC}} = 1.4626 - 1.4475 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right) \\ - 0.0382 \cdot (1 - HHI)^2 \cdot \left(1 - \overline{\beta}\right) + 0.3289 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right)^2 \end{aligned} \tag{5.41}$$

[303] We have tried several different regressions but similar to Cespedes et al. (2006), this function worked best. In contrast to Cespedes et al. (2006) we do not set the first parameter a_0 to one because our DF-factor is not bounded by the single-factor model.

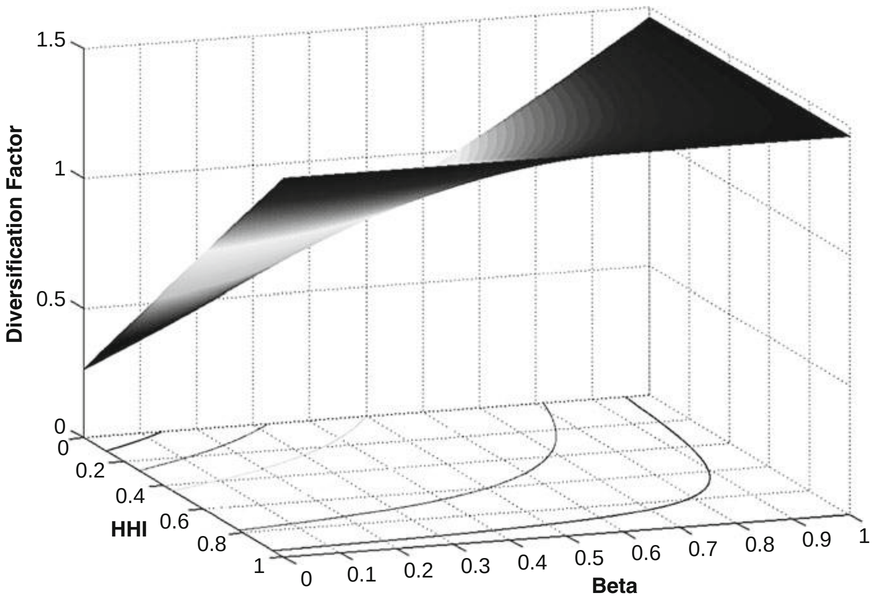

Fig. 5.2 Surface plot of the *DF*-function

with $R^2 = 95.5\%$. Analogously, we determined the *DF*-function when using the Pykhtin formula

$$DF_{\text{Pykhtin}} = 1.4598 - 1.4168 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right) - 0.0213 \cdot (1 - HHI)^2 \cdot \left(1 - \overline{\beta}\right) + 0.2421 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right)^2 \tag{5.42}$$

with a coefficient of determination of $R^2 = 97.9\%$. The latter function is plotted in Fig. 5.2.[304] To finally get the approximation for the multi-factor model, (5.39) has to be computed using either function (5.41) or (5.42).

It can be seen that the maximum diversification factor is about 1.46. Thus, in the case of (almost) no diversification effects, the measured capital requirement is 46% above the regulatory capital under Pillar 1. This will appear in the case of being concentrated to a single sector, leading to $HHI = 1$, as well as in the theoretical case

[304] The shape of the function is similar to Cespedes et al. (2006) but their range is from 0.1 to 1.0 whereas our function ranges from 0.2 to 1.5. In addition, they received a little higher R^2 (99.4% instead of 95.5% or 97.9%) but this is mainly due to the different simulation setting. Cespedes et al. (2006) directly draw the parameter $\overline{\beta}$ as an input parameter for each simulation, implying $\overline{\beta}$ to fully define their correlation structure. We use a heterogeneous correlation structure instead and compute $\overline{\beta}$ for the portfolios. Thus, in our setting $\overline{\beta}$ does not reflect the complete correlation structure, which results in a lower R^2 but does not imply a worse approximation.

of perfect correlations between the relevant sectors, leading to $\bar{\beta} = 1$. Furthermore, the diversification factor is strongly increasing in *HHI* and in $\bar{\beta}$, which is consistent with the intuition.

5.2.4 Accounting for Sector Concentrations with the Model of Düllmann

5.2.4.1 The Binomial Expansion Technique

The model of Düllmann (2006) is a combination of the Binomial Expansion Technique (BET)-model and the Infection Model of Davis and Lo (2001). For this reason, at first, the BET-model and the infection model will be explained, before the model of Düllmann will be presented and applied to our multi-factor setting. During the application, we will deviate from the original procedure in order to apply the ES instead of the VaR and to accelerate the computation time significantly for large portfolios.[305]

The *Binomial Expansion Technique* (BET) was developed by Moody's for the rating of CDOs but it can also be applied to standard credit portfolios without tranches. The BET-model approximates the loss distribution of the portfolio but is much less computationally intensive than Monte Carlo simulations.[306] The main idea is to perform a mapping of the original portfolio into a hypothetical homogeneous portfolio with stochastically independent, Bernoulli distributed loss events leading to a binomial distributed number of losses. The hypothetical portfolio can be described by the average probability of default $\overline{p}$, the number of credits D, which is called the modified Diversity Score, and the (constant) Loss Given Default LGD. The parameters D and $\overline{p}$ are calibrated in a way that the first two moments of the original and the hypothetical portfolio loss distribution are identical. This shall lead to a similar overall loss distribution of both portfolios.

With n_s for the number of credits in sector s, the loss of the original portfolio equals

$$\tilde{L}^{\text{orig}} = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD \cdot 1_{\{\tilde{D}_{s,i}\}}, \tag{5.43}$$

whereas the loss of the hypothetical portfolio is

$$\tilde{L}^{\text{hyp}} = \sum_{i=1}^{D} w \cdot LGD \cdot 1_{\{\tilde{D}_i\}} = \sum_{i=1}^{D} \frac{1}{D} \cdot LGD \cdot 1_{\{\tilde{D}_i\}}. \tag{5.44}$$

[305] In comparison to the original procedure, the computation time could be reduced by almost 99.9% in our calculations.

[306] Cf. Cifuentes et al. (1996), Cifuentes and O'Connor (1996), and Cifuentes and Wilcox (1998).

Matching the expectation for both portfolios leads to[307]

$$\overline{p} := \mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) = \sum_{s=1}^{S}\sum_{i=1}^{n_s} w_{s,i} \cdot PD_{s,i} \tag{5.45}$$

and matching the variance results in[308]

$$D = \frac{\overline{p} \cdot (1-\overline{p})}{\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i} \cdot w_{t,j} \cdot \mathrm{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}. \tag{5.46}$$

The pairwise default correlation and the asset correlation between borrower i in sector s and borrower j in sector t can be transformed into each other with[309]

$$\mathrm{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right) = \frac{\Phi_2\left(\Phi^{-1}(PD_i), \Phi^{-1}\left(PD_j\right), \mathrm{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right)\right) - PD_i \cdot PD_j}{\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}. \tag{5.47}$$

In the original model, it is assumed that the correlation between every two borrowers, which are in the same sector, is identical. Furthermore, it is assumed that the correlation between two borrowers in distinct sectors is always identical and the PDs inside a sector are homogeneous. These assumptions would lead to some simplifications in (5.45)–(5.47), but they are not necessary for the calculation of the loss distribution. Thus, we can also use the correlation structure from (5.4) and use (5.45)–(5.47). Having determined the parameters $\overline{p}$ and D, we can calculate the loss distribution for the hypothetical portfolio. Since the (uncertain) number of defaults $\tilde{k}$ in the hypothetical portfolio is binomially distributed

$$\tilde{k} = \sum_{i=1}^{D} 1_{\{\tilde{D}_i\}} \sim \mathcal{B}(D, \overline{p}), \tag{5.48}$$

the probability of having k defaults is

$$P_k = \mathbb{P}\left(\tilde{k} = k\right) = \mathbb{P}\left(\sum_{i=1}^{D} 1_{\{\tilde{D}_i\}} = k\right) = \binom{D}{k} \cdot (\overline{p})^k \cdot (1-\overline{p})^{D-k}. \tag{5.49}$$

[307] See Appendix 5.5.5.

[308] See Appendix 5.5.5.

[309] See Appendix 5.5.6.

The corresponding cumulative distribution function for the number of defaults is

$$F_k(x) = \mathbb{P}\left(\tilde{k} \le x\right) = \mathbb{P}\left(\sum_{i=1}^{D} 1_{\{\tilde{D}_i\}} \le x\right) = \sum_{k=0}^{x} P_k. \tag{5.50}$$

Thus, the loss distribution of the original portfolio can be approximated with

$$\begin{aligned} F_{\text{orig}}^{(n)}(l) \approx F_{\text{hyp}}^{(D)}(l) &= \mathbb{P}\left(\sum_{i=1}^{D} \frac{1}{D} \cdot LGD \cdot 1_{\{\tilde{D}_i\}} \le l\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}} \le \frac{l \cdot D}{LGD}\right) = \sum_{k=0}^{\lfloor l \cdot D/LGD \rfloor} P_k, \end{aligned} \tag{5.51}$$

leading to a VaR of

$$VaR_\alpha\left(\tilde{L}^{\text{orig}}\right) \approx VaR_\alpha\left(\tilde{L}^{\text{hyp}}\right) = \frac{1}{D} \cdot LGD \cdot F_k^{-1}(\alpha), \tag{5.52}$$

where $F_k^{-1}(\alpha)$ is the inverse CDF of the binomial distribution with parameters D and $\overline{p}$ from (5.48). The ES can be computed using the definition of the ES (2.19). From (5.48) and (5.52) it can best be seen that the interaction between the credits is incorporated by reducing the real number of credits to the hypothetical number, the Diversity Score, with higher exposure weights. E.g., for $D = n/2$, each (stochastically independent) default in the hypothetical portfolio is equivalent to two defaults in the original portfolio, which leads to some kind of default interaction in the original portfolio.

5.2.4.2 The Infectious Defaults Model

Davis and Lo (2001) present an alternative to the BET-model for the determination of the loss distribution of a credit portfolio which is assumed to be homogeneous.[310] In the model, credits can default not only directly but they can also be "*infected*" by other credits leading to an indirect default. Similar to the BET-model, the direct defaults are assumed to be stochastically independent, leading to a binomial distribution of direct defaults. Thus, the task is how the indirect defaults can be incorporated into the loss distribution. To begin with, several indicator variables are introduced, which indicate the type of default and the interaction. Whether a credit defaults or not is expressed by the indicator variable $\tilde{Z}_i$, which equals one in the event of default and zero otherwise. Thus, the total number of defaults in the portfolios is

$$\tilde{k} = \tilde{Z}_1 + \tilde{Z}_2 + \cdots + \tilde{Z}_n. \tag{5.53}$$

[310]Similar to the BET-model, the authors developed their model for CDOs but it can also be applied to standard credit portfolios.

If credit i defaults directly, the indicator variable $\tilde{X}_i$ takes the value one. Furthermore, the indicator variable $\tilde{Y}_{j,i}$ indicates whether credit j could *potentially* infect credit i. The condition for this infection is that both the infection variable $\tilde{Y}_{j,i}$ and the direct default indicator $\tilde{X}_j$ of credit j take the value one. This leads to the following function for the default indicator $\tilde{Z}_i$:

$$\tilde{Z}_i = \tilde{X}_i + \left(1 - \tilde{X}_i\right) \cdot \left(1 - \prod_{j \neq i} \left(1 - \tilde{X}_j \cdot \tilde{Y}_{j,i}\right)\right), \text{with}\, i = 1, ..., n \quad \text{and} \quad j = 1, ..., n. \tag{5.54}$$

In (5.54), the second term is only relevant if credit i does not default directly. In this case, an infection through any one or several credits leads to a product of zero so that the second term equals one. The equation will be demonstrated further with the following examples for a portfolio consisting of four credits:

- Credit 1 defaults directly:

$$\begin{aligned} Z_1 &= X_1 + (1 - X_1) \cdot \left(1 - \prod_{j \neq i} \left(1 - X_j \cdot Y_{j,i}\right)\right) \\ &= 1 + (1 - 1) \cdot \left(1 - \prod_{j \neq i} \left(1 - X_j \cdot Y_{j,i}\right)\right) = 1. \end{aligned}$$

  As the term $\left(1 - X_1\right)$ equals zero, the last expression vanishes and Credit 1 defaults directly without an effect of defaults of the other credits.

- Credit 2 defaults as a consequence of infection from the defaulted credit 1:

$$\begin{aligned} Z_2 &= X_2 + (1 - X_2) \cdot \left(1 - \left(1 - X_1 \cdot Y_{1,2}\right) \cdot \left(1 - X_3 \cdot Y_{3,2}\right) \cdot \left(1 - X_4 \cdot Y_{4,2}\right)\right) \\ &= 0 + (1 - 0) \cdot (1 - (1 - 1 \cdot 1) \cdot (1 - 0 \cdot 1) \cdot (1 - 1 \cdot 0)) = 1. \end{aligned}$$

  The non-defaulting Credit 3 would also have the potential to infect Credit 2 in the case of a default. Credit 4 defaults but does not infect credit 2.

- Credit 3 does not default:

$$\begin{aligned} Z_3 &= X_3 + (1 - X_3) \cdot \left(1 - \left(1 - X_1 \cdot Y_{1,3}\right) \cdot \left(1 - X_2 \cdot Y_{2,3}\right) \cdot \left(1 - X_4 \cdot Y_{4,3}\right)\right) \\ &= 0 + (1 - 0) \cdot (1 - (1 - 1 \cdot 0) \cdot (1 - 0 \cdot 0) \cdot (1 - 1 \cdot 0)) = 1. \end{aligned}$$

  The third credit does neither default directly nor indirectly.

  In a probabilistic setting, a direct default is assumed to happen with probability p:

$$\mathbb{P}\left(\tilde{X}_i = 1\right) = p \quad \forall i. \tag{5.55}$$

  Similar, the infection indicator $\tilde{Y}_{j,i}$ takes the value one with probability q:

$$\mathbb{P}\left(\tilde{Y}_{j,i} = 1\right) = q \quad \forall i, j. \tag{5.56}$$

Thus, the dependence structure is assumed to be perfectly homogeneous. Let i be the number of direct defaults, k–i the number of indirect defaults, so that we have in total k defaults, and the other n–k credits do not default. The probability of observing k defaults out of n credits is

$$P_k = \binom{n}{k} \cdot \sum_{i=1}^{k} \binom{k}{i} \cdot \underbrace{p^i}_{i \text{ direct defaults}} \cdot \underbrace{\left[(1-p) \cdot \left(1-(1-q)^i\right)\right]^{k-i}}_{k-i \text{ indirect defaults}} \cdot \underbrace{\left[(1-p) \cdot (1-q)^i\right]^{n-k}}_{n-k \text{ survivors}} . \tag{5.57}$$

The probability P_k can be split into four parts:

(a) If we ignore the perturbations, the probability of i direct defaults is p^i.
(b) A number of k–i indirect defaults occurs if these credits do not default directly, which has the probability $(1-p)^{k-i}$, but these are infected by any of the i directly defaulted credits with probability $(1-(1-q)^i)^{k-i}$.
(c) For a survival of n–k credits, these credits default neither directly, which has a probability of $(1-p)^{n-k}$, nor any of the i directly defaulted credits leads to an indirect default, which can be expressed as $((1-q)^i)^{n-k}$.
(d) There are several possible perturbations of defaulted credits. Firstly, there are $\binom{n}{k}$ perturbations for k out of n defaults. Furthermore, there are several combinations of direct and indirect defaults. A number of k defaults can consist of $(1;\ k-1)$, $(2;\ k-2)$, ..., $(k;\ 0)$ direct and indirect defaults. For each of these combinations, there exist $\binom{k}{i}$ perturbations. All of the corresponding probabilities have to be summed up to cover all combinations for k defaults.

Expression (5.57) could also be written as

$$\begin{aligned}
P_k &= \binom{n}{k} \cdot \sum_{i=1}^{k} \binom{k}{i} \cdot p^i \cdot (1-p)^{n-i} \cdot \left(1-(1-q)^i\right)^{k-i} \cdot (1-q)^{i(n-k)} \\
&= \binom{n}{k} \cdot \sum_{i=1}^{k-1} \binom{k}{i} \cdot p^i \cdot (1-p)^{n-i} \cdot \left(1-(1-q)^i\right)^{k-i} \cdot (1-q)^{i(n-k)} \\
&\quad + \binom{n}{k} \cdot \binom{k}{k} \cdot p^k \cdot (1-p)^{n-k} \cdot \left(1-(1-q)^k\right)^0 \cdot (1-q)^{k(n-k)} \qquad (5.58) \\
&= \binom{n}{k} \cdot \Bigg[p^k \cdot (1-p)^{n-k} \cdot (1-q)^{k(n-k)} \\
&\quad + \sum_{i=1}^{k-1} \binom{k}{i} \cdot p^i \cdot (1-p)^{n-i} \cdot \left(1-(1-q)^i\right)^{k-i} \cdot (1-q)^{i(n-k)}\Bigg],
\end{aligned}$$

which corresponds to the original formula of Davis and Lo (2001). Thus, with the *Infectious Defaults Model* (IDM), we obtain the following distribution of defaults:

$$F^{\mathrm{IDM}}(x) = \sum_{k=0}^{x} P_k, \tag{5.59}$$

or, in analogy to (5.51), we obtain the loss distribution

$$F^{\mathrm{IDM}}(l) = \sum_{k=0}^{\lfloor l \cdot n / LGD \rfloor} P_k. \tag{5.60}$$

The VaR can be calculated as

$$VaR_{\alpha}^{\mathrm{IDM}}(\tilde{L}) = \left(F_{\alpha}^{\mathrm{IDM}}\right)^{-1}(l) = \frac{1}{n} \cdot LGD \cdot \left(F_{\alpha}^{\mathrm{IDM}}\right)^{-1}(x), \tag{5.61}$$

and the ES can be computed using the definition of the ES (2.19).

The main problem for an application of (5.60) is to determine the probability of a direct default p and the infection probability q. Usually, statistical models only provide the (combined) probability of default PD without separating these types of defaults. Thus, if the infection probability q could be determined exogenously, it is plausible to demand that the probability p shall be consistent with the estimation of PD with respect to the expected number of defaults:[311]

$$\mathbb{E}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}}\right) = n \cdot \left(1 - (1-p) \cdot (1 - p \cdot q)^{n-1}\right) \stackrel{!}{=} n \cdot PD. \tag{5.62}$$

Consequently, the remaining task is to find a method to estimate q from historical or market data. Unfortunately, this problem could not be solved by Davis and Lo (2001). Thus, for the time being it seems necessary to rely on the opinion of experts which infection probabilities seem to be reasonable for a specific portfolio or sector.

5.2.4.3 Integrating Infectious Defaults into the BET-Model

Setup of the Model

As demonstrated by Düllmann (2006), the BET-model can significantly underestimate the VaR if the asset returns of the credits are positively correlated. Thus, the BET-model seems not suitable for measuring concentration risk, which is usually characterized by a high degree of default interaction. Against this background, Düllmann (2006) combines the infection model of Davis and Lo

[311] The expected number of defaults in the infectious defaults model is determined in Appendix 5.5.7.

(2001), which explicitly considers default interaction, with the BET-model. For this purpose, at first a heterogeneous portfolio is mapped into a comparable homogeneous portfolio as in the BET-model. Thus, the average probability of default $\overline{p}$ as well as the Diversity Score D are calculated according to (5.45) and (5.46). Using this hypothetical portfolio consisting of D credits, the default distribution is calculated on the basis of the infectious defaults model, leading to the following expression for the VaR in the *infection model* (IM) of Düllmann:[312]

$$VaR_{\alpha}^{\mathrm{IM}}\left(\tilde{L}\right) = \frac{1}{D} \cdot LGD \cdot \left(F_{\alpha}^{\mathrm{IDM}}\right)^{-1}(x), \tag{5.63}$$

with the distribution function $F_{\alpha}^{\mathrm{IDM}}$ of the infectious defaults model from (5.59). At this point, the probabilities of a direct default p and indirect default q are still required as additional input parameters. Similar to the suggestion of Davis and Lo (2001) to choose the parameter p for a given parameter q in a way that the expected loss is correct, Düllmann (2006) proposes to choose the parameters in a way that the VaR is identical to the "true" VaR of a multi-factor model. For this purpose, he determines the VaR at confidence level 0.999 with Monte Carlo simulations and chooses the parameter q for a given parameter p that solves the following equation:

$$VaR_{0.999}^{\mathrm{IM}}\left(\tilde{L}\right) \stackrel{!}{=} VaR_{0.999}^{\mathrm{MC}}\left(\tilde{L}\right). \tag{5.64}$$

In principle, it is possible not only to match the VaR but also to match the EL. In this case, both parameters p and q would be a result of these two conditions. Instead, Düllmann (2006) uses only condition (5.64) and uses the value of the averaged PD for the parameter p. Since the direct defaults should actually be only a part of the total number of defaults, the expectation of the loss distribution is too high when using this approach. However, if only the VaR is of interest, this procedure should be sufficient.[313]

The next steps are very similar to the procedure of Cespedes et al. (2006). At first, the VaR is computed for a multitude of portfolios and the corresponding infection probabilities q are determined. Then, the infection probability is explained by several portfolio variables with a linear regression. For this purpose, Düllmann (2006) chooses the following regression model:

$$\ln(q) = a_0 + a_1 \cdot \ln(HHI) + a_2 \cdot \ln(\overline{p}) + a_3 \cdot \ln(\overline{r}_{\mathrm{Intra}}) + a_4 \cdot \ln(\overline{r}_{\mathrm{Inter}}) + \varepsilon, \tag{5.65}$$

where the explanatory variables shall explain most of the dependence structure. The Herfindahl–Hirschmann index HHI is calculated as the sum of squared relative exposure shares of the sectors in the portfolio, which is similar to the definition used by Cespedes et al., who rely on the share of regulatory Pillar 1 capital instead of the

[312] Cf. (5.61) for the corresponding expression without using the parameters of the BET-model.

[313] Düllmann (2006) mentions that the simultaneous computation of both parameters leads to numerical problems. For this reason, the discrepancy in the EL is accepted. Cf. Düllmann (2006), p. 10.

share of exposure. The average probability of default $\overline{p}$ is calculated with (5.45). The variables $\overline{r}_{\text{Intra}}$ and $\overline{r}_{\text{Inter}}$ are the average intra- and inter-sector correlations, which are weighted by the total exposure amounts of the corresponding sectors. Thus, the calculation is similar to the average weighted inter-sector correlation $\overline{\beta}$ from (5.37), except for the weighting with the total exposure instead of the regulatory capital under Pillar 1. In this context, it is important to notice that Düllmann (2006) uses a definition of the sector correlations that is slightly different from the definition used in the preceding sections. While we use the term inter-sector correlation for the correlation between the sector factors, Düllmann (2006) uses this expression for the correlation between the asset returns of two borrowers, which belong to different sectors, leading to

$$\operatorname{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right) = \begin{cases} 1 & \text{if } s = t \text{ and } i = j, \\ \overline{r}_{\text{Intra}} & \text{if } s = t \text{ and } i \neq j, \\ \overline{r}_{\text{Inter}} & \text{if } s \neq t, \end{cases} \tag{5.66}$$

which already takes into account that the correlation parameters are assumed to be homogeneous. Thus, the relation between "our" intra- and inter-sector correlation ρ_{Intra} and ρ_{Inter} and "Düllmann's" correlation parameters $\overline{r}_{\text{Intra}}$ and $\overline{r}_{\text{Inter}}$ is

$$\overline{r}_{\text{Intra}} = \rho_{\text{Intra}} \quad \text{and} \quad \overline{r}_{\text{Inter}} = \rho_{\text{Intra}} \cdot \rho_{\text{Inter}} \tag{5.67}$$

in a homogeneous setting.[314] The coefficients $a_0, \ldots, a_4$ of regression model (5.65) are estimated using the ordinary least squares (OLS) technique. Finally, after application of the resulting regression function, the VaR can be approximated for any credit portfolio by computation of (5.63).

Calibration and Implementation of the Model

For the calibration of the model, several portfolios are constructed which differ in the degree of concentration, the PDs, and the correlation coefficients.[315] It is assumed that the portfolio consists of 2,000 credits with identical exposure size. In the first of four portfolio types there are only three different sectors with a sectoral exposure weight of 50%, 30%, and 20%. This leads to a HHI of 38%. The second portfolio is constructed from the first one by splitting each sector into two new sectors, where the first one has a share of 2/3 and the second one of 1/3. The same procedure is repeated for the third and the fourth portfolio type, so that the last portfolio consists of $3 \cdot 2^3 = 24$ sectors and contains the smallest sector concentration with a HHI of 6.5%. In addition to this variation, the probability of default is

[314] See also definition (5.4) of Sect. 5.2.1.

[315] The portfolios used for calibration correspond to the setting of Düllmann (2006).

Table 5.4 Parameter combinations for the calibration of the model

PD (%)	$\bar{r}_{\text{Intra}}(\%)$	$\bar{r}_{\text{Inter}}(\%)$		
0.03	5.0	2.5		
0.20	10.0	2.5	5.0	
0.50	15.0	2.5	5.0	7.5
1.00	20.0	5.0	7.5	10.0
2.00	30.0	5.0	10.0	15.0
5.00	40.0	5.0	10.0	15.0

varied between 0.03% and 5%, the correlation parameter $\bar{r}_{\text{Intra}}$ between 5% and 40%, and the correlation parameter $\bar{r}_{\text{Inter}}$ between 2.5% and 15%.[316] These parameters are identical for every credit of a specific portfolio. Thus, for each of the four mentioned portfolio types the parameter combinations shown in Table 5.4 are applied, leading to 360 portfolios in total.

Consistent with the preceding sections, we implement the ES instead of the VaR. Thus, for each of these portfolios, the ES is computed on the basis of a standard Monte Carlo simulation. Within the calculation of ES in the infection model, the value of the averaged PD is used for the parameter p as noticed before. Then, the infection probability q is determined, which leads to a match between the ES of the infection model and the Monte Carlo simulation:

$$ES_{0.999}^{\text{IM}}\left(\tilde{L}\right) \stackrel{!}{=} ES_{0.999}^{\text{MC}}\left(\tilde{L}\right). \tag{5.68}$$

When determining the ES in the infection model, we have to calculate the inverse CDF $(F_{\alpha}^{\text{IDM}})^{-1}$ with (5.59) and the Diversity Score D with (5.46), which requires the default correlation of (5.47). The computation of D can be quite time-consuming but the calculation can be accelerated significantly. Looking at the Diversity Score

$$D = \frac{\overline{p}\cdot(1-\overline{p})}{\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \text{Corr}\left(1_{\{\tilde{D}_{s,i}\}},1_{\{\tilde{D}_{t,j}\}}\right)\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}, \tag{5.69}$$

we find that the calculation requires n^2-times the calculation of the denominator, with $\sum_{s=1}^{S} n_s = n$ for the total number of credits, and especially n^2-times the calculation of the default correlation.[317] Similar to the computation of the multi-factor adjustment from Sect. 5.2.2.2, building PD-classes for each sector can reduce the calculation time notably. With N_{PD} for the number of PD-classes, we can build $S\cdot N_{PD} =: B$ different buckets with a number of n_u credits in each bucket u

[316] Due to the characteristic of the correlation parameter (5.67), the parameter $\bar{r}_{\text{Inter}}$ is always smaller than the parameter $\bar{r}_{\text{Intra}}$.

[317] See (5.47).

$(\sum_{u=1}^{B} n_u = n)$. Thus, a bucket u corresponds to all credits in a specific combination of a sector and a PD-class. Using this notation, the denominator of D can be written as

$$\begin{aligned}\frac{\overline{p}\cdot(1-\overline{p})}{D} &= \sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \mathrm{Corr}\left(1_{\{\tilde{D}_{s,i}\}},1_{\{\tilde{D}_{t,j}\}}\right)\\ &\quad\cdot\sqrt{PD_{s,i}(1-PD_{s,i})PD_{t,j}(1-PD_{t,j})}\\ &= \sum_{u=1}^{B}\sum_{v=1}^{B} w_u\cdot w_v\cdot \mathrm{Corr}\left(1_{\{\tilde{D}_u\}},1_{\{\tilde{D}_v\}}\right)\cdot\sqrt{PD_u(1-PD_u)PD_v(1-PD_v)}\\ &\quad+\sum_{u=1}^{B}\sum_{i=1}^{n_B} {w_{u,i}}^2\cdot\left(1-\mathrm{Corr}\left(1_{\{\tilde{D}_u\}},1_{\{\tilde{D}_u\}}\right)\right)\cdot PD_u\cdot(1-PD_u).\end{aligned} \tag{5.70}$$

The first term of the resulting expression utilizes that the default correlation between creditors and the PDs are identical within each bucket. Therefore, we can sum up the corresponding terms. However, this term neglects that the asset correlation $\mathrm{Corr}(\tilde{a}_{s,i},\tilde{a}_{s,i})$ of a credit with itself equals one. Instead, these elements are treated as if the correlation was $\mathrm{Corr}(\tilde{a}_{s,i},\tilde{a}_{s,i}) \equiv \overline{r}_{\text{Intra}}$, which is only true for $i \neq j$. Thus, we have to exchange the corresponding default correlations and set the correlation to one. This is done in the second fraction. Obviously, the computation time of (5.70) is now predominated by:[318]

$$\text{Loops} = B^2 = (N_{\text{PD}}\cdot S)^2. \tag{5.71}$$

Corresponding to the finding for the Pykhtin model, the number of loops does not grow with bigger portfolios. Thus, it is possible to compute the formula on bucket level within reasonable time.[319]

Using these terms, we determine the required infection probability q. As a next step, for all 360 portfolios the explanatory variables of the regression model (5.65) are calculated and the OLS-regression is performed. This leads to the following estimation function for q:

$$\begin{aligned}\ln(q) &= 0.8467 + 0.5017\cdot\ln(HHI) + 0.4726\cdot\ln(\overline{p})\\ &\quad+1.0849\cdot\ln(\overline{r}_{\text{Intra}}) + 0.6782\cdot\ln(\overline{r}_{\text{Inter}}),\end{aligned} \tag{5.72}$$

[318]For the second fraction, a number of n elements has to be computed. Depending on the number of buckets or credits, the computation time can be longer than for the first term, but due to the linearity this term is virtually unproblematic.

[319]The computation time when calculating the infection model on bucket- instead on borrower-level can be reduced from 12 min to less than 1 s for a portfolio with 11 sectors, 7 PD-classes, and 5,000 creditors.

with a coefficient of determination of $R^2 = 96.7\%$. Using this formula, the infection probability and herewith the ES of every credit portfolio can be approximated very fast. If the portfolio is heterogeneous, the input parameters $\overline{p}$, $\overline{r}_{\text{Intra}}$, and $\overline{r}_{\text{Inter}}$ are the weighted averages instead of the individual parameters as described in the previous section. The performance of this model as well as the performance of the models presented in Sects. 5.2.2 and 5.2.3 will be analyzed subsequently.

5.3 Performance of Multi-Factor Models

5.3.1 Analysis for Deterministic Portfolios

To determine the quality of the presented models, we start our analysis with calculating the risk for five deterministic portfolios of different quality.[320] We generate well-diversified portfolios consisting of 5,000 credits. Consequently, we have neither high name nor high sector concentration risk. For this, we choose the sectors and their weights as given in Table 5.2. The inter-sector correlation is given in Table 5.1 whereas the intra-sector correlation is calculated on the basis of (5.8). The five portfolios differ in their PD distribution which is presented in Fig. 4.7. Portfolio 1 is the portfolio with the highest and Portfolio 5 is the one with the lowest credit quality distribution.

In Table 5.5, we compare the results from the Monte Carlo simulations (MC-Sim.), the Basel II formula (Basel II), the multi-factor adjustment of Pykhtin (Pykhtin), the formula that is based on Cespedes et al. (2006) if Monte Carlo simulations are used for calibration (CHKR I) or if the Pykhtin formula is used for the calibration (CHKR II), and the infection model of Düllmann (Düllmann). The results from the Monte Carlo simulations using the risk measure ES serve as the benchmark for the other models.

As can be seen in the table, the benchmark portfolio is constructed in a way that the Basel II formula represents a very good approximation[321] of the “real” ES in a multi-factor model given by Monte Carlo simulations.[322] Besides, the simulated VaR^{mf} matches the simulated ES^{mf}, our benchmark, almost exactly. The calculated values of the Pykhtin model are very good approximations of the ES in almost all cases, too. The outcomes of the CHKR model are somewhat more imprecise in both cases. With better credit quality, the estimation error is

[320]The results refer to the total gross loss of a portfolio in terms of ES or VaR. To relate this to the unexpected net loss, the results have to be multiplied by the LGD and the EL has to be subtracted.

[321]The small mismatch is mainly due to keeping the ES-confidence level constant and not a result of the chosen intra-sector correlation function. If we directly compare the results from Monte Carlo simulations with the ES in the ASRF framework, the relative root mean squared error is reduced from 0.97% to 0.28%.

[322]In our analyses, the number of simulation runs is 500,000.

Table 5.5 Comparison of the models for the five benchmark portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES (%)	6.23	7.68	12.95	20.88	23.15
	VaR (%)	6.18	7.62	12.94	20.93	23.30
	Absolute error (bp)	−5	−6	−1	5	15
	Relative error (%)	−0.80	−0.78	0.08	0.24	0.65
Basel II	VaR (%)	6.12	7.59	12.95	20.89	23.26
	Absolute error (bp)	−11	−9	0	1	11
	Relative error (%)	−1.77	−1.17	0.00	0.05	0.48
Pykhtin	ES (%)	6.21	7.66	12.91	20.80	23.20
	Absolute error (bp)	−2	−2	−4	−8	5
	Relative error (%)	−0.32	−0.26	−0.31	−0.38	0.22
CHKR I	ES (%)	6.07	7.51	12.70	20.43	22.79
	Absolute error (bp)	−16	−17	−25	−45	−36
	Relative error (%)	−2.57	−2.21	−1.93	−2.16	−1.56
CHKR II	ES (%)	6.00	7.45	12.68	20.48	22.87
	Absolute error (bp)	−23	−23	−27	−40	−28
	Relative error (%)	−3.69	−2.99	−2.08	−1.92	−1.21
Düllmann	ES (%)	6.86	8.87	15.42	23.29	25.95
	Absolute error (bp)	63	119	247	241	280
	Relative error (%)	10.19	15.49	19.06	11.54	12.07

increasing, which leads to an underestimation of risk in high quality portfolios. However, the infection model of Düllmann shows a rather poor performance for all benchmark portfolios and overestimates the true ES significantly.

As a next step, we change the portfolio structure towards high sector concentration. For this purpose, we increase the sector weights of two sectors. We assume that 45% of the creditors – in terms of their exposure – belong to the Information Technology sector and an equal amount belongs to the Telecommunication Services sector. The remaining 10% of exposure are equally assigned to the miscellaneous sectors. As shown in Table 5.6, the risk materially increases for all types of portfolio quality. Again, the simulated values for ES^{mf} and VaR^{mf} are very close to each other. However, the Basel formula underestimates the risk by 14–20%, depending on the portfolio quality. This is the (relative) amount that should be considered in the assessment of capital adequacy under Pillar 2. The approximation formula of Pykhtin can capture this concentration risk with a negligible error in all cases. CHKR I leads to an underestimation of risk in high quality portfolios and to an overestimation of risk in low quality portfolios with a maximum deviation of nearly 4%. By contrast, in most cases the model CHKR II underestimates the risk with at maximum 6%. Thus, the sector concentration risk is not fully captured for high quality portfolios. The model of Düllmann fails to approximate the true risk and leads to a material overestimation of risk.

Furthermore, we built credit portfolios with low sector concentration. For this purpose, we use the concept of naïve diversification, implying each sector to have an equal weight of 1/11. As can be seen in Table 5.7, the economic capital is significantly lower than the regulatory capital. Moreover, this shows that it is easy

Table 5.6 Comparison of the models for five high concentrated portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES (%)	7.69	9.22	15.41	24.41	27.10
	VaR (%)	7.48	9.17	15.36	24.51	27.06
	Absolute error (bp)	–21	–5	–5	10	–6
	Relative error (%)	–2.73	–0.54	–0.32	0.41	0.15
Basel II	VaR (%)	6.12	7.59	12.95	20.89	23.26
	Absolute error (bp)	–157	–163	–246	–352	–384
	Relative error (%)	–20.42	–17.68	–15.96	–14.42	–14.17
Pykhtin	ES (%)	7.66	9.29	15.46	24.39	27.03
	Absolute error (bp)	–3	7	5	–2	–7
	Relative error (%)	–0.35	0.76	0.31	–0.08	–0.24
CHKR I	ES (%)	7.40	9.08	15.59	25.07	27.95
	Absolute error (bp)	–29	–14	18	66	85
	Relative error (%)	–3.77	1.52	1.17	2.70	3.14
CHKR II	ES (%)	7.22	8.86	15.19	24.38	27.14
	Absolute error (bp)	–47	–36	–22	–3	4
	Relative error (%)	–6.11	–3.90	–1.43	–0.12	0.15
Düllmann	ES (%)	8.97	11.30	19.77	28.26	31.21
	Absolute error (bp)	128	208	436	385	411
	Relative error (%)	16.60	22.52	28.27	15.77	15.17

Table 5.7 Comparison of the models for five low concentrated portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES (%)	5.66	6.98	12.16	19.78	22.06
	VaR (%)	5.64	6.94	12.17	19.81	22.10
	Absolute error (bp)	−2	–4	1	3	4
	Relative error (%)	–0.35	–0.57	0.08	0.15	0.18
Basel II	VaR (%)	6.12	7.59	12.95	20.89	23.26
	Absolute error (bp)	46	61	79	111	120
	Relative error (%)	8.13	8.74	6.50	5.61	5.44
Pykhtin	ES (%)	5.67	6.98	12.14	19.74	22.08
	Absolute error (bp)	1	0	–2	–4	2
	Relative error (%)	0.26	–0.07	–0.16	–0.21	0.09
CHKR I	ES (%)	5.66	6.94	11.92	19.17	21.38
	Absolute error (bp)	0	–4	–24	–61	–68
	Relative error (%)	0.0	–0.57	–1.97	–3.08	–3.08
CHKR II	ES (%)	5.64	6.94	12.06	19.52	21.81
	Absolute error (bp)	–2	–4	–10	–26	–25
	Relative error (%)	–0.35	–0.57	–0.82	–1.31	–1.13
Düllmann	ES (%)	5.93	7.46	13.52	21.07	23.58
	Absolute error (bp)	27	48	136	129	152
	Relative error (%)	4.71	6.95	11.19	6.51	6.90

to construct portfolios that are better diversified than the overall credit market.[323] Apart from insignificant deviations, both simulated risk measures lead to the same solutions. Again, the Pykhtin model approximates the "real" risk very good for all types of credit quality. The CHKR model I underestimates the risk for high quality portfolios with up to 3%. The CHKR model II underestimates the risk, too, but the approximation error is negligible. Again, the model of Düllmann overestimates the true risk and leads to a similar performance as the Basel II model.

5.3.2 Simulation Study for Homogeneous and Heterogeneous Portfolios

To achieve more general results, we test the models for different, randomly generated portfolios. For this reason, we implement four simulation studies. In these studies, we analyze the accuracy for homogeneous as well as for heterogeneous portfolios with respect to PD and EAD. In each simulation run, we generate a portfolio and determine its ES by the different models. After 100 runs, we calculate the root mean squared error for the outcomes of the Pykhtin model, the CHKR models I and II,[324] and the model of Düllmann in absolute and relative terms to quantify the performance of the models in comparison to Monte Carlo simulations using 500,000 trials. Furthermore, we calculate the VaR with the Basel II formula and with a Monte Carlo simulation to measure its accuracy compared to ES^{mf}. In the following, we describe the four simulation settings.

Simulation I. In this scenario, we generate portfolios with homogenous exposure sizes and homogenous PDs, that is, $w_i = 1/5000$ and $PD_i = PD = \mathrm{const}$ for each credit. To test the accuracy for different portfolio qualities, a PD is drawn from a uniformly distribution between 0 and 10% before each new run. The sector structure and correlation is the same as in Sect. 5.2.1.

Simulation II. We generate portfolios with homogenous exposure sizes but heterogeneous PDs. For each sector, we randomly determine one of the quality distributions from Fig. 4.7. After that, we draw the PD for each credit of the sector according to this quality distribution. The exposure size remains as in Simulation I. Again, the sector structure and correlation is taken from Sect. 5.2.1.

Simulation III. We generate portfolios with homogenous PDs as in Simulation I but with heterogeneous exposure sizes. Firstly, we randomly choose the number of sectors between 2 and 11. Then, we apply a uniform distribution between 0 and 1 for the weight of every sector and scale this such that the weights sum up to one. The weights for the credits in each sector are determined in the same manner. The correlations remain unchanged.

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[323]If we consider all 25,000 simulated portfolios from Sect. 5.2.3, the lowest measured economic capital requirement was even 26% lower than the regulatory capital. This underlines the prospects of actively managing credit portfolios, e.g. with credit derivatives, but this is not in the scope of this thesis.

[324]CHKR I still corresponds to the *DF*-function based on Monte Carlo simulation and CHKR II on the Pykhtin formula.

Simulation IV. In this setting, the PDs as well as the exposure sizes of the generated portfolios are heterogeneous. The PDs are determined as in Simulation II and the exposure sizes as in Simulation III.

In each simulation, we calculate the intra-sector correlations with (5.8) and choose 5,000 credits. These portfolios contain a relatively low amount of name concentration. Instead, we focus on sector concentration. The reason is that the identical methodology for measuring name concentrations, the granularity adjustment, can be used within all implemented approaches. Thus, we prefer to avoid name concentrations to be able to separately analyze the effect of sector concentrations. The degree of sector concentration differs between the simulations. In Simulation I and II, the portfolios consist of homogenous exposures, so their HHI is in each case $1/11 = 9.1\%$. This equals the value for a naïve diversified portfolio. On the contrary, in Simulation III and IV exposures are chosen randomly and the HHI of the generated portfolios can take values between 9.1% (naïve diversification) and 1 (perfect concentration). The mean of these HHIs is around 30% in each simulation, which is only slightly higher than the HHIs of the bank portfolios analyzed by Acharya et al. (2006), which shows that the setting leads to a realistic degree of diversification.[325] The results of our simulation study can be found in Table 5.8.

Again, the outcomes of the Pykhtin model are good approximations of the "true" ES calculated with Monte Carlo simulations. Especially, if EADs are heterogeneous (simulation setting III and IV), the results are very good. Both types of the CHKR

Table 5.8 Accuracy of different models in comparison with the "true" ES calculated with Monte Carlo simulations for the specified simulation studies

		Simulation Setting I	Simulation Setting II	Simulation Setting III	Simulation Setting IV
MC-Sim. VaR	Ø Absolute error (bp)	18	6	22	8
	Ø Relative error (%)	0.67	0.43	0.77	0.60
Basel II	Ø Absolute error (bp)	259	186	264	379
	Ø Relative error (%)	11.66	13.70	8.81	25.76
Pykhtin	Ø Absolute error (bp)	14	11	54	18
	Ø Relative error (%)	0.64	0.81	3.40	1.26
CHKR I	Ø Absolute error (bp)	54	11	47	20
	Ø Relative error (%)	1.73	0.79	1.65	1.53
CHKR II	Ø Absolute error (bp)	54	12	46	21
	Ø Relative error (%)	1.72	0.84	1.56	1.59
Düllmann	Ø Absolute error (bp)	103	185	139	224
	Ø Relative error (%)	5.84	8.58	5.84	11.28

[325] Acharya et al. (2006) examined credit portfolios of 105 Italian banks during the period 1993–1999. In this study, most bank portfolios had a HHI between 20% and 30%. However, it has to be considered that the number of different industry sectors was 23 whereas we use 11 different sectors. Thus, for a comparable degree of diversification their calculated HHI have to be slightly smaller than our HHIs.

model lead to very stable results in all simulation settings. Interestingly, the CHKR model performs even better if PDs are heterogeneous, probably because the portfolios used for calculation of the functional form have heterogeneous PDs, too, and thus the resulting portfolios are more similar. It is somewhat surprising that in Simulation III the CHKR model shows a better performance than the Pykhtin model, even if the Pykhtin formula is used for determination of the diversification factor. Probably, the approximation errors of the Pykhtin model are partially smoothed by the regression from (5.40). The results of the Düllmann model are not convincing. The model can generate better outcomes than the Basel II model but performs materially worse than the other presented models. A reason could be that the portfolios which were used for the calibration of the model are too different from the portfolios of the simulation study. Against this background, it could be interesting to repeat the calibration procedure which has been applied to the CHKR model instead of the procedure presented in Sect. 5.2.4.3 because these calibration portfolios are very similar to those used in the simulation study. Of course, this calibration would be much more time-consuming than the applied calibration if we use all 25,000 randomly generated portfolios of the CHKR calibration instead of the 360 deterministic portfolios suggested by Düllmann (2006).

The comparison of the risk measures with different confidence levels shows an almost perfect match between ES^{mf} and VaR^{mf}. The relative error is smaller than 1% in each case, so our simulation study clarifies that the above-mentioned theoretical problems of the VaR are not practically relevant for a very broad range of credit portfolios. Hence, there is nothing to be said against the use of the VaR for determining the credit risk from a practical point of view even if the portfolio incorporates sector concentration risk. The Basel formula, however, shows the largest inaccuracy of all tested models for any simulation. Since in simulation setting I and II a naïve diversified portfolio is taken as a basis, the Basel formula overestimates the risk in every case due to the diversification effect.

A plot of the relative errors of the Basel formula and of VaR^{mf} in simulation setting III, sorted in ascending order, can be found in Fig. 5.3. Apart from slightly higher deviations, a plot with a similar characteristics results for simulation setting IV. It can be seen that for more than 50% of the simulated portfolios the Basel VaR is too low. That means the risk measured under Pillar 1 is underestimated compared to the “real” risk. In general, this happens when the sector concentration of the generated portfolio increases, as already demonstrated for deterministic portfolios. Thus, the simulation study accentuates the need for considering sector concentration when calculating the risk of a credit portfolio. Otherwise, the risk can be massively underestimated. This conclusion coincides with that of BCBS (2006), which points out that sector concentration can increase the capital requirement up to 40%. The maximal deviation of VaR^{mf} is around 3%. Actually, for most of the generated portfolios the error is almost zero. Thus, the deviation is negligible for practical implementation. Nevertheless, in order to verify whether there is a systematic pattern, which may help to explain the occurrence of these deviations in the multi-factor setting, we have tried to find portfolio variables such as *HHI*, average correlation, or average *PD* that can explain these deviations. Since our analyses

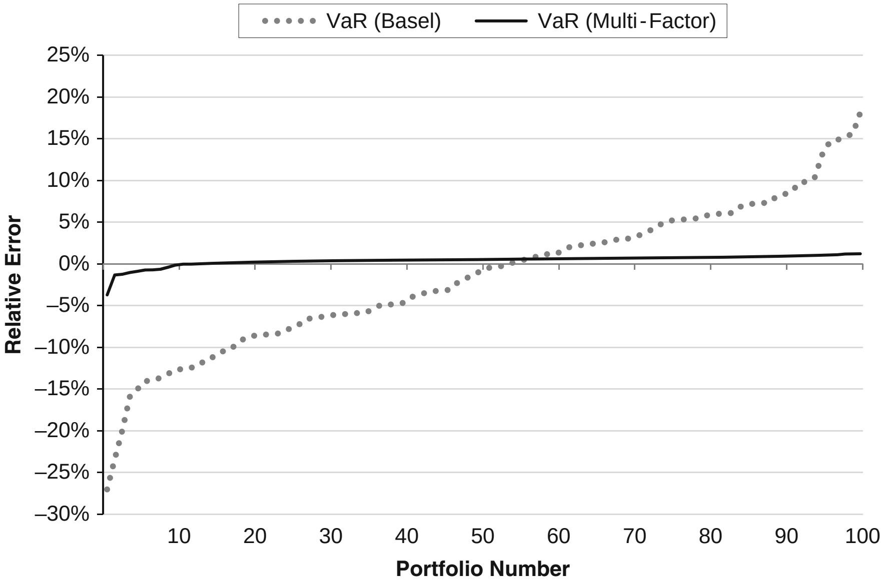

Fig. 5.3 Deviations of VaR^{Basel} and VaR^{mf} from ES^{mf}

Table 5.9 Comparison of the runtime

	Runtime: Calibration	Runtime: Application
MC-Simulation		20 min
Pykhtin		~5 s–2 min
CHKR I	30 days	0.01 s
CHKR II	150 min	0.01 s
Düllmann	240 min	~1–10 s

did not show a link between the deviations and any of the mentioned variables, it seems that the occurrence is unsystematic.

As the purpose of deriving (semi-)analytical approximation formulas for the VaR or the ES is an acceleration of the computation time, we compare the runtime of the demonstrated methods in Table 5.9.[326]

The main advantage of the Pykhtin model is that it can be applied without an excessive calibration procedure and that it is considerably faster than Monte Carlo simulations without leading to major approximation errors. The advantage of the Düllmann model is that its application is much faster but this comes at the cost of a higher approximation error. When comparing both alternative implementations of the CHKR model, we strongly propose to use the Pykhtin model for calibration (CHKR II) instead of Monte Carlo simulations (CHKR I), as the approximation

[326]The runtimes refer to a quad-core PC with 2.66 GHz CPUs (calculated on one core).

accuracy is almost identical but the computation time for determination of the *DF*-function is significantly lower. As this calibration procedure has to be computed only once for a specified correlation structure and the application of the formula is very fast, in most situations the CHKR type model should be a very good choice.

5.4 Interim Result

In this chapter, we have proposed a methodology to perform multi-factor models that are able to measure concentration risk in credit portfolios in terms of economic capital. In contrast to the existing literature regarding concentration risk, this procedure delivers results that are consistent with Basel II and has the advantage of quite low data requirements since the intra-sector correlation does not have to be estimated from historical bank data. Furthermore, we have applied this methodology to different multi-factor approaches. Since the calibration or application of these approaches is quite time-consuming for large portfolios in the original settings, which is one of the main problems of these approaches, we have demonstrated how these calculations can be accelerated significantly. As a next step, we have compared the performance of these approaches within a simulation study as the capability of different models to measure sector concentration risk has only been tested in a rather brief analysis of Düllmann (2007) before. It could be shown that it is possible to achieve good approximations in reasonable time if the approaches are adjusted in the proposed way. We have also analyzed whether the theoretical shortcomings of the Value at Risk, which can arise when leaving the ASRF framework, lead to undesirable results. Although it is indisputable that the ES has theoretical advantages over the VaR, which has already been demonstrated in several contrived portfolio examples, our framework seems well suited to explore this question for a variety of more realistic credit portfolios. We find that the accuracy of the VaR turns out to be almost perfect compared to the ES for a multitude of generated portfolios. Therefore, in our opinion, it is unproblematic to use the VaR for measuring sector concentration risk in credit portfolios.

During the specification of the multi-factor setting, we have determined input parameters, especially the inter- and intra-sector correlations, in a way that the results are comparable with the regulatory Pillar 1 capital. Thus, we do not follow some approaches that assume a pure diversification effect compared with the Basel II formula. Instead, we relate the results to a well-diversified portfolio as assumed when calibrating the Basel II formula and determine a function for the implied intra-sector correlation. Hence, it is possible to directly consider the extent of credit risk concentrations in the assessment of capital adequacy under Pillar 2. Using these modifications, we have performed an extensive numerical study similar to Cespedes et al. (2006) to get a closed form approximation formula and show how the calibration can be accelerated significantly without worsening the accuracy. In addition, we suggest computing the multi-factor adjustment and the infection model on a bucket instead of a borrower level. This allows computing the formulas of Pykhtin (2004) as well as the formulas of Düllmann (2006) much faster than Monte Carlo simulations even for a high number of credits. Moreover, due to the theoretical advantages of ES, we have determined the approximation formulas for our modified variants of Cespedes et al. (2006) and Düllmann (2006) using the risk measure ES instead of the VaR.

Having assured Basel II consistent capital requirements, we have analyzed the impact of credit concentration risk and have carried out a simulation study to compare the performance of the (modified) models from Pykhtin (2004), Cespedes et al. (2006), and Düllmann (2006). We find that the Pykhtin model leads to very good results for homogeneous as well as heterogeneous PDs if EADs are homogeneous. The performance is slightly lower for heterogeneous EADs. The results of the Cespedes-type model have a throughout high accuracy. Interestingly, the approach works better for heterogeneous portfolios. In comparison, the model of Düllmann (2006) performs rather poorly. In general, the models of Pykhtin (2004) as well as the Cespedes-type model are both well-suited for approximating the economic capital in a multi-factor setting when adjusted in the proposed way. The main advantage of the Pykhtin model is that it can directly be applied to an arbitrary portfolio type, whereas the Cespedes-type approach should not be used without initially performing the demonstrated extensive numerical work if the portfolio structure is very different. On the contrary, the results of the Cespedes-type model are slightly better for heterogeneous portfolios and it allows for ad-hoc analyses including sensitivity analyses when the non-recurring extensive numerical work is progressed.

5.5 Appendix

5.5.1 *Optimal Choice of the Single Correlation Factor*

To relate $\tilde{\bar{L}}$ to $\tilde{L}$, it is assumed that the new systematic factor $\tilde{\bar{x}}$ has a linear dependence to the original sector factors:[327]

$$\tilde{\bar{x}} = \sum_{k=1}^{K} b_k \cdot \tilde{z}_k, \tag{5.73}$$

$$\text{with} \quad \sum_{k=1}^{K} b_k^2 = 1. \tag{5.74}$$

Condition (5.74) satisfies that the new systematic factor still has a variance of 1. In order to specify the correlation factors c_i and the coefficients b_k, it will be required that the loss $\tilde{\bar{L}}$ equals the conditional expectation of the "true" loss $\mathbb{E}(\tilde{L}|\tilde{\bar{x}})$.

[327]In contrast to this representation, Pykhtin (2004) applies these and the following formulas to n sector factors whereas we use K sector factors with $K \leqslant n$. This can lead to a significant reduction of the computation time as will be shown later on.

This assures that the first element of the subsequently performed Taylor series expansion vanishes.[328] To determine $\mathbb{E}(\tilde{\tilde{L}}|\tilde{\tilde{x}})$, we first recall that the asset return of obligor i in sector s can be written as

$$\tilde{a}_{s,i} = \sqrt{\rho_{\text{Intra},i}} \cdot \tilde{x}_s + \sqrt{1 - \rho_{\text{Intra},i}} \cdot \tilde{\xi}_i. \tag{5.75}$$

Now, each original sector factor $\tilde{x}_s$ is decomposed into a part that is related to the single-factor $\tilde{\tilde{x}}$ and a part that is independent of this factor:

$$\tilde{x}_s = \overline{\rho}_s \cdot \tilde{\tilde{x}} + \sqrt{1 - \overline{\rho}_s^{\,2}} \cdot \tilde{\eta}_s, \tag{5.76}$$

with $\tilde{\eta}_s \sim \mathcal{N}(0,1)$. Using (5.2), (5.73), and the independence of $\tilde{z}_i,\ \tilde{z}_j$ if $i \neq j$, the correlation parameter $\overline{\rho}_s$ can be expressed as

$$\begin{aligned} \overline{\rho}_s = \operatorname{Corr}\left(\tilde{x}_s, \tilde{\tilde{x}}\right) &= \operatorname{Corr}\left(\sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k, \sum_{k=1}^{K} b_k \cdot \tilde{z}_k\right) \\ &= \sum_{k=1}^{K} \alpha_{s,k} \cdot b_k \cdot \mathbb{V}(\tilde{z}_k) = \sum_{k=1}^{K} \alpha_{s,k} \cdot b_k. \end{aligned} \tag{5.77}$$

Using this notation, the asset return (5.75) can now be written as

$$\begin{aligned} \tilde{a}_{s,i} &= \sqrt{\rho_{\text{Intra},i}} \cdot \tilde{x}_s + \sqrt{1 - \rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\ &= \sqrt{\rho_{\text{Intra},i}} \cdot \left(\overline{\rho}_s \cdot \tilde{\tilde{x}} + \sqrt{1 - \overline{\rho}_s^{\,2}} \cdot \tilde{\eta}_s\right) + \sqrt{1 - \rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\ &= \sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_s \cdot \tilde{\tilde{x}} + \sqrt{\rho_{\text{Intra},i} - \rho_{\text{Intra},i} \cdot \overline{\rho}_s^{\,2}} \cdot \tilde{\eta}_s + \sqrt{1 - \rho_{\text{Intra},i}} \cdot \tilde{\xi}_i. \end{aligned} \tag{5.78}$$

The independent standard normally distributed random variables $\tilde{\eta}_s$ and $\tilde{\xi}_i$ can be combined into a new standard normally distributed random variable $\tilde{\zeta}_i$, leading to

$$\tilde{a}_{s,i} = \sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i \cdot \tilde{\tilde{x}} + \sqrt{1 - \left(\sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i\right)^2} \cdot \tilde{\zeta}_i, \tag{5.79}$$

with $\overline{\rho}_i = \overline{\rho}_s$ for each obligor i in sector s. Since the variable $\tilde{\zeta}_i$ is independent of $\tilde{\tilde{x}}$, we can use the known formula of the single-factor model for the conditional expectation

$$\mathbb{E}\left(\tilde{L}|\tilde{\tilde{x}}\right) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \Phi\left[\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i \cdot \Phi^{-1}(\alpha)}{\sqrt{1 - \left(\sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i\right)^2}}\right]. \tag{5.80}$$

[328] This simplification of the Taylor series could already be used for the granularity adjustment in Sect. 4.2.1.1.

The mentioned condition $\tilde{\tilde{L}} = \mathbb{E}(\tilde{L}|\tilde{\tilde{x}})$ leads to

$$\begin{aligned}
&\tilde{\tilde{L}} = \mathbb{E}(\tilde{L}|\tilde{\tilde{x}}) \\
&\Leftrightarrow \Phi\left[\frac{\Phi^{-1}(PD_i) - c_i \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i{}^2}}\right] = \Phi\left[\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i \cdot \Phi^{-1}(\alpha)}{\sqrt{1-\left(\sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i\right)^2}}\right] \\
&\Leftrightarrow c_i = \sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i = \sqrt{\rho_{\text{Intra},i}} \cdot \sum_{k=1}^{K} \alpha_{i,k} \cdot b_k,
\end{aligned} \tag{5.81}$$

using (5.9), (5.80), (5.77), and $\alpha_{i,k} = \alpha_{s,k}$ for each obligor i in sector s. While $\rho_{\text{Intra},i}$ and $\alpha_{i,k}$ are known, the coefficients b_k are unknown.

While (5.81) already satisfies that the first-order term of the Taylor series vanishes, the concrete choice of the parameter set $\{b_k\}$ is critical concerning the distance between the zeroth-order term $q_\alpha(\tilde{\tilde{L}})$ and the unknown quantile $q_\alpha(\tilde{L})$. Unfortunately, it is not obvious how this distance can be minimized. Thus, Pykhtin (2004) relies on the intuition that coefficients which maximize the (weighted) correlation between the single factor $\tilde{\tilde{x}}$ and the sector factors $\{\tilde{x}_s\}$ should lead to good results. This leads to the following maximization problem:

$$\max_{\{b_k\}}\left(\sum_{i=1}^{n} d_i \cdot \overline{\rho}_i\right) = \max_{\{b_k\}}\left(\sum_{i=1}^{n} d_i \cdot \sum_{k=1}^{K} \alpha_{i,k} \cdot b_k\right), \tag{5.82}$$

subject to

$$\sum_{k=1}^{K} b_k^2 = 1. \tag{5.83}$$

The solution of this optimization problem is[329]

$$b_k = \sum_{i=1}^{n} \frac{d_i \cdot \alpha_{ik}}{2\tau}, \tag{5.84}$$

where the positive constant Lagrange multiplier τ is chosen in a way that $\{b_k\}$ satisfies the constraint. As a final step, the weighting factor d_i has to be chosen. After trying several specifications, Pykhtin (2004) uses

$$d_i = w_i \cdot LGD_i \cdot \Phi\left[\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_{\text{Intra},i}} \cdot \Phi^{-1}(\alpha)}{\sqrt{1-\rho_{\text{Intra},i}}}\right], \tag{5.85}$$

[329] Cf. Pykhtin (2004).

which is the VaR formula in a single-factor model. The intuition behind this choice is that obligors with a high exposure in terms of VaR should have a large weight in the maximization problem whereas obligors with a small VaR should have a minor impact. Summing up, the correlation parameter c_i results from (5.81), where the coefficients b_k are determined by (5.83)–(5.85).

5.5.2 Conditional Correlation

The correlation conditional on $\tilde{\bar{x}}$ between the asset returns from (5.19) can be written as

$$\begin{aligned}\rho_{ij}^{\bar{x}} &= \operatorname{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j} \middle| \tilde{\bar{x}}\right)\\ &= \frac{\operatorname{Cov}\left(\sum_{k=1}^{K}\left(\sqrt{\rho_{\text{Intra},i}}\cdot\alpha_{s,k} - c_i\cdot b_k\right)\cdot\tilde{z}_k, \sum_{k=1}^{K}\left(\sqrt{\rho_{\text{Intra},j}}\cdot\alpha_{t,k} - c_j\cdot b_k\right)\cdot\tilde{z}_k\right)}{\sqrt{\mathbb{V}\left(\tilde{a}_{s,i}|\tilde{\bar{x}}\right)}\cdot\sqrt{\mathbb{V}\left(\tilde{a}_{t,j}|\tilde{\bar{x}}\right)}}\\ &= \frac{\sum_{k=1}^{K}\left(\sqrt{\rho_{\text{Intra},i}}\cdot\alpha_{s,k} - c_i\cdot b_k\right)\cdot\left(\sqrt{\rho_{\text{Intra},j}}\cdot\alpha_{t,k} - c_j\cdot b_k\right)}{\sqrt{1-c_i{}^2}\cdot\sqrt{1-c_j{}^2}},\end{aligned} \tag{5.86}$$

using the independence of the factors $\tilde{z}_k$. The numerator can be simplified using $\sum_{k=1}^{K}\alpha_{s,k}\cdot b_k = c_i\big/\sqrt{\rho_{\text{Intra},i}}$ from (5.81) and $\sum_{k=1}^{K} b_k^2 = 1$ from (5.74):

$$\begin{aligned}&\sum_{k=1}^{K}\left(\sqrt{\rho_{\text{Intra},i}}\cdot\alpha_{s,k} - c_i\cdot b_k\right)\cdot\left(\sqrt{\rho_{\text{Intra},j}}\cdot\alpha_{t,k} - c_j\cdot b_k\right)\\ &\quad= \sqrt{\rho_{\text{Intra},i}}\cdot\sqrt{\rho_{\text{Intra},j}}\cdot\sum_{k=1}^{K}\alpha_{s,k}\cdot\alpha_{t,k} - \sqrt{\rho_{\text{Intra},i}}\cdot c_j\cdot\sum_{k=1}^{K}\alpha_{s,k}\cdot b_k\\ &\qquad - \sqrt{\rho_{\text{Intra},j}}\cdot c_i\cdot\sum_{k=1}^{K}\alpha_{t,k}\cdot b_k + c_i\cdot c_j\cdot\sum_{k=1}^{K} b_k{}^2\\ &\quad= \sqrt{\rho_{\text{Intra},i}}\cdot\sqrt{\rho_{\text{Intra},j}}\cdot\sum_{k=1}^{K}\alpha_{s,k}\cdot\alpha_{t,k} - \sqrt{\rho_{\text{Intra},i}}\cdot c_j\cdot\frac{c_i}{\sqrt{\rho_{\text{Intra},i}}}\\ &\qquad - \sqrt{\rho_{\text{Intra},j}}\cdot c_i\cdot\frac{c_j}{\sqrt{\rho_{\text{Intra},j}}} + c_i\cdot c_j\\ &\quad= \sqrt{\rho_{\text{Intra},i}}\cdot\sqrt{\rho_{\text{Intra},j}}\cdot\sum_{k=1}^{K}\alpha_{s,k}\cdot\alpha_{t,k} - c_j\cdot c_i.\end{aligned} \tag{5.87}$$

This leads to

$$\rho_{ij}^{\bar{x}} = \frac{\sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} - c_i \cdot c_j}{\sqrt{1 - c_i{}^2} \cdot \sqrt{1 - c_j{}^2}}. \tag{5.88}$$

5.5.3 Calculation of the Decomposed Variance

In order to determine the conditional variance, it is decomposed into the following terms:[330]

$$\mathbb{V}\left(\tilde{L}|\tilde{\bar{x}} = \bar{x}\right) = \mathbb{V}\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\bar{x}} = \bar{x}\right] + \mathbb{E}\left[\mathbb{V}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\bar{x}} = \bar{x}\right]. \tag{5.89}$$

For calculation of these terms, first the expressions (a) $\mathbb{E}(\tilde{L}|\{\tilde{z}_k\})$, (b) $\mathbb{E}(\tilde{L}^2|\{\tilde{z}_k\})$, and (c) $\mathbb{V}(\tilde{L}|\{\tilde{z}_k\})$ will be calculated. The conditional loss is given as

$$\tilde{L}|\{\tilde{z}_k\} = \sum_i w_i \cdot \left(\widetilde{LGD}_i|\{\tilde{z}_k\}\right) \cdot \left(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right), \tag{5.90}$$

and for stochastically independent LGDs this leads to

$$\tilde{L}|\{\tilde{z}_k\} = \sum_i w_i \cdot \widetilde{LGD}_i \cdot \left(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right). \tag{5.91}$$

(a) With $\mathbb{E}\left(\widetilde{LGD}_i\right) =: ELGD_i$ and $\mathbb{E}\left(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right) =: p_i(\{\tilde{z}_k\})$ we obtain:

$$\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right) = \sum_i w_i \cdot ELGD_i \cdot p_i(\{\tilde{z}_k\}). \tag{5.92}$$

(b) Consider that $1^2_{\{\tilde{D}_i\}} = 1_{\{\tilde{D}_i\}}$, $\mathbb{E}\left(\widetilde{LGD}^2\right) = \mathbb{E}^2\left(\widetilde{LGD}\right) + \mathbb{V}\left(\widetilde{LGD}\right) =: ELGD^2 + VLGD$, and

$$\begin{aligned} \mathbb{E}\left(LGD_i LGD_j\right) &= \text{Cov}\left(LGD_i, LGD_j\right) + \mathbb{E}(LGD_i)\mathbb{E}\left(LGD_j\right) \\ &= \mathbb{E}(LGD_i)\mathbb{E}\left(LGD_j\right) \\ &=: ELGD_i ELGD_j, \end{aligned} \tag{5.93}$$

[330] The following calculations are based on Tasche (2006a), p. 41 ff.

as well as

$$\begin{aligned}\mathbb{E}\left(1_{\{\tilde{D}_i\}}1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\right) &= \mathrm{Cov}\left(1_{\{\tilde{D}_i\}}, 1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\right) + \mathbb{E}\left(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right)\mathbb{E}\left(1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\right)\\ &= \mathbb{E}\left(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right)\mathbb{E}\left(1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\right)\\ &= p_i(\{\tilde{z}_k\})p_j(\{\tilde{z}_k\}).\end{aligned} \tag{5.94}$$

Moreover, we have

$$\left(\sum_i x_i\right)^2 = \sum_i\sum_j x_ix_j = \sum_i x_i{}^2 + \sum_i\sum_{j\neq i} x_ix_j, \tag{5.95}$$

$$\sum_{j\neq i} x_ix_j = \sum_j x_ix_j - x_i{}^2. \tag{5.96}$$

Thus, we obtain:

$$\begin{aligned}\mathbb{E}\left(\tilde{L}^2|\{\tilde{z}_k\}\right) &= \mathbb{E}\left[\sum_i \left(w_i\widetilde{LGD}_i 1_{\{\tilde{D}_i\}}\right)^2|\{\tilde{z}_k\}\right]\\ &= \mathbb{E}\left[\sum_i w_i{}^2\widetilde{LGD}_i{}^2 1^2_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right]\\ &\qquad + \mathbb{E}\left[\sum_i\sum_{j\neq i} w_iw_j\widetilde{LGD}_i\widetilde{LGD}_j 1_{\{\tilde{D}_i\}}1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\right]\\ &= \sum_i w_i{}^2\mathbb{E}\left(LGD_i{}^2\right)p_i(\{\tilde{z}_k\})\\ &\quad + \sum_i\sum_{j\neq i} w_iw_j\mathbb{E}\left(LGD_iLGD_j\right)\mathbb{E}\left(1_{\{\tilde{D}_i\}}1_{\{\tilde{D}_j\}}\right)\\ &= \sum_i w_i{}^2\left(ELGD_i{}^2 + VLGD_i\right)p_i(\{\tilde{z}_k\})\\ &\quad + \sum_i\sum_{j\neq i} w_iw_jELGD_iELGD_jp_i(\{\tilde{z}_k\})p_j(\{\tilde{z}_k\})\\ &= \sum_i w_i{}^2\left(ELGD_i{}^2 + VLGD_i\right)p_i(\{\tilde{z}_k\}) - \sum_i w_i{}^2ELGD_i{}^2p_i{}^2(\{\tilde{z}_k\})\\ &\quad + \sum_i\sum_j w_iw_jELGD_iELGD_jp_i(\{\tilde{z}_k\})p_j(\{\tilde{z}_k\})\\ &= \sum_i w_i{}^2\left(ELGD_i{}^2\left[p_i(\{\tilde{z}_k\}) - p_i{}^2(\{\tilde{z}_k\})\right] + VLGD_ip_i(\{\tilde{z}_k\})\right)\\ &\quad + \mathbb{E}^2\left(\tilde{L}|\{\tilde{z}_k\}\right).\end{aligned} \tag{5.97}$$

(c) The conditional variance $\mathbb{V}(\tilde{L}|\{\tilde{z}_k\})$ is equal to

$$\begin{aligned}\mathbb{V}\big(\tilde{L}|\{\tilde{z}_k\}\big) &= \mathbb{E}\Big(\tilde{L}^2|\{\tilde{z}_k\}\Big) - \mathbb{E}^2\big(\tilde{L}|\{\tilde{z}_k\}\big)\\ &= \sum_i {w_i}^2\big(ELGD_i{}^2\big[p_i(\{\tilde{z}_k\}) - p_i{}^2(\{\tilde{z}_k\})\big] + VLGD_i p_i(\{\tilde{z}_k\})\big).\end{aligned} \tag{5.98}$$

(d) Using the law of iterated expectation, we have

$$p_i(\overline{x}) = \mathbb{E}\Big(1_{\{\tilde{D}_i\}}|\tilde{\overline{x}} = \overline{x}\Big) = \mathbb{E}\Big[\mathbb{E}\Big(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\Big)|\overline{x}\Big] = \mathbb{E}[p_i(\{\tilde{z}_k\})|\overline{x}]. \tag{5.99}$$

Thus, with (5.98) the expectation of the conditional variance can be written as

$$\begin{aligned}\mathbb{E}\big[\mathbb{V}\big(\tilde{L}|\{\tilde{z}_k\}\big)|\tilde{\overline{x}}=\overline{x}\big] &= \sum_i {w_i}^2\big(ELGD_i{}^2\big(\mathbb{E}[p_i(\{\tilde{z}_k\})|\overline{x}] - \mathbb{E}\big[p_i{}^2(\{\tilde{z}_k\})|\overline{x}\big]\big)\\ &\quad +VLGD_i\mathbb{E}[p_i(\{\tilde{z}_k\})|\overline{x}]\big)\\ &= \sum_i {w_i}^2\Big(ELGD_i{}^2\Big(p_i(\overline{x}) - \mathbb{P}\Big[\Big(1_{\{\tilde{D}_i\}} = 1\Big) \wedge \Big(1_{\{\tilde{D}_i{}'\}} = 1\Big)|\overline{x}\Big]\Big)\\ &\quad +VLGD_i p_i(\overline{x})\big).\end{aligned} \tag{5.100}$$

For independent idiosyncratic factors $\tilde{\zeta}_i, \tilde{\zeta}_i{}' \sim \mathcal{N}(0,1)$ and with

$$p_i(\overline{x}) := \Phi\left(\frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-{c_i}^2}}\right) \Leftrightarrow \frac{\Phi^{-1}(PD) - c_i\cdot\overline{x}}{\sqrt{1-{c_i}^2}} = \Phi^{-1}(p_i(\overline{x})), \tag{5.101}$$

we get

$$\begin{aligned}&\mathbb{P}\Big[\Big(1_{\{\tilde{D}_i\}} = 1\Big) \wedge \Big(1_{\{\tilde{D}_i{}'\}} = 1\Big)|\overline{x}\Big]\\ &\quad = \mathbb{P}\Big[c_i\cdot\tilde{\overline{x}} + \sqrt{1-{c_i}^2}\cdot\tilde{\zeta}_i \le \Phi^{-1}(PD_i), c_i\cdot\tilde{x} + \sqrt{1-{c_i}^2}\cdot\tilde{\zeta}_i{}' \le \Phi^{-1}(PD_i)|\overline{x}\Big]\\ &\quad = \mathbb{P}\left[\tilde{\zeta}_i \le \frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-{c_i}^2}}, \tilde{\zeta}_i{}' \le \frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-{c_i}^2}}\right]\\ &\quad = \Phi_2\big(\Phi^{-1}(p_i(\overline{x})), \Phi^{-1}(p_i(\overline{x})), \rho_{ii}^{\overline{x}}\big),\end{aligned} \tag{5.102}$$

with the correlation conditional on $\overline{x}$ of (5.20). Hence, (5.100) results in

$$\mathbb{E}\left[\mathbb{V}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\overline{x}}=\overline{x}\right]=\sum_i w_i^{\,2}\left(ELGD_i^{\,2}\left[p_i(\overline{x})-\Phi_2\left(\Phi^{-1}(p_i(\overline{x})),\Phi^{-1}(p_i(\overline{x})),\rho_{ii}^{\overline{x}}\right)\right]\right.$$
$$\left.+VLGD_i p_i(\overline{x})\right). \tag{5.103}$$

(e) Using (5.92), the variance of the conditional expectation can be expressed as

$$\begin{aligned}
&\mathbb{V}\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\overline{x}}=x\right]\\
&=\mathbb{E}\left[\mathbb{E}^2\left(\tilde{L}|\{\tilde{z}_k\}\right)|\overline{x}\right]-\mathbb{E}^2\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\overline{x}\right]\\
&=\mathbb{E}\left(\mathbb{E}^2\left[\sum_i w_i\widetilde{LGD}_i 1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right]|\overline{x}\right)-\mathbb{E}^2\left(\sum_i w_i ELGD_i p_i(\{\tilde{z}_k\})|\overline{x}\right)\\
&=\mathbb{E}\left[\left(\sum_i w_i ELGD_i p_i(\{\tilde{z}_k\})\right)^2|\overline{x}\right]-\left(\sum_i w_i ELGD_i p_i(\overline{x})\right)^2,
\end{aligned} \tag{5.104}$$

leading to

$$\begin{aligned}
&\mathbb{V}\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\overline{x}}=x\right]\\
&=\mathbb{E}\left[\sum_i\sum_j w_i w_j ELGD_i ELGD_j p_i(\{\tilde{z}_k\})\cdot p_j(\{\tilde{z}_k\})|\overline{x}\right]\\
&\quad-\sum_i\sum_j w_i w_j ELGD_i ELGD_j p_i(\overline{x})p_j(\overline{x})\\
&=\sum_i\sum_j w_i w_j ELGD_i ELGD_j \mathbb{E}\left(p_i(\{\tilde{z}_k\})\cdot p_j(\{\tilde{z}_k\})|\overline{x}\right)\\
&\quad-\sum_i\sum_j w_i w_j ELGD_i ELGD_j p_i(\overline{x})p_j(\overline{x})\\
&=\sum_i\sum_j w_i w_j ELGD_i ELGD_j\left[\mathbb{P}\left[\left(1_{\{\tilde{D}_i\}}=1\right)\wedge\left(1_{\{\tilde{D}_j\}}=1\right)|\overline{x}\right]-p_i(\overline{x})p_j(\overline{x})\right].
\end{aligned} \tag{5.105}$$

Analogous to (5.102) and using the conditional correlation (5.20), this can be expressed as:

$$\mathbb{V}\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\overline{x}}=\overline{x}\right]=\sum_i\sum_j w_i w_j ELGD_i ELGD_j$$
$$\cdot\left[\Phi_2\left(\Phi^{-1}(p_i(\overline{x})),\Phi^{-1}\left(p_j(\overline{x})\right),\rho_{ij}^{\overline{x}}\right)-p_i(\overline{x})p_j(\overline{x})\right]. \tag{5.106}$$

5.5.4 Derivatives of the Decomposed Variance Terms

As both conditional variance terms are linear in the bivariate normal distribution, the derivative of the bivariate normal distribution will be calculated subsequently. Then, the derivatives of $\eta_{2,c}^{\infty}(\bar{x})$ and $\eta_{2,c}^{\mathrm{GA}}(\bar{x})$ will be computed.

Proposition. *The derivative of the bivariate normal distribution can be written as:*

$$\frac{d}{dx}\Phi_2\left(\Phi^{-1}(p_i(\bar{x})),\Phi^{-1}\left(p_j(\bar{x})\right),\rho_{ij}^{\bar{x}}\right)=\frac{dp_i(\bar{x})}{d\bar{x}}\Phi\left(\frac{\Phi^{-1}\left(p_j(\bar{x})\right)-\rho_{ij}^{\bar{x}}\cdot\Phi^{-1}(p_i(\bar{x}))}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)$$
$$+\frac{dp_j(\bar{x})}{d\bar{x}}\Phi\left(\frac{\Phi^{-1}(p_i(\bar{x}))-\rho_{ij}^{\bar{x}}\cdot\Phi^{-1}\left(p_j(\bar{x})\right)}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right). \tag{5.107}$$

Proof. Using the notation

$$y_i(\bar{x})=\frac{\Phi^{-1}(PD_i)-c_i\cdot\bar{x}}{\sqrt{1-c_i{}^2}},\ \ y_j(\bar{x})=\frac{\Phi^{-1}(PD_j)-c_j\cdot\bar{x}}{\sqrt{1-c_j{}^2}}, \tag{5.108}$$

and the chain rule, we get

$$\begin{aligned}&\frac{d}{d\bar{x}}\Phi_2\left(\Phi^{-1}(p_i(\bar{x})),\Phi^{-1}\left(p_j(\bar{x})\right),\rho_{ij}^{\bar{x}}\right)\\&=\frac{d}{d\bar{x}}\Phi_2\left(y_i(\bar{x}),y_j(\bar{x}),\rho_{ij}^{\bar{x}}\right)\\&=\underbrace{\frac{dy_i}{d\bar{x}}}_{(I)}\underbrace{\frac{\partial}{\partial y_i}\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)}_{(II)}+\underbrace{\frac{dy_j}{d\bar{x}}}_{(III)}\underbrace{\frac{\partial}{\partial y_j}\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)}_{(IV)}.\end{aligned} \tag{5.109}$$

For calculation of term (II) and (IV), we rewrite the bivariate normal distribution according to Appendix 2.8.6 as

$$\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)=\int_{z=-\infty}^{y_j}\varphi(z)\Phi\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)dz. \tag{5.110}$$

Thus, we have

$$\begin{aligned}
\frac{\partial}{\partial y_i}\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right) &= \frac{\partial}{\partial y_i}\int\limits_{z=-\infty}^{y_j}\varphi(z)\Phi\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)dz \\
&= \frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\int\limits_{z=-\infty}^{y_j}\varphi(z)\varphi\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)dz \\
&= \frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\int\limits_{z=-\infty}^{y_j}\frac{1}{2\pi}\exp\left(-\frac{1}{2}\underbrace{\left[z^2+\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)^2\right]}_{(*)}\right)dz.
\end{aligned} \tag{5.111}$$

The term (*) is equivalent to

$$\begin{aligned}
z^2+\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)^2 &= \frac{\left(1-\left(\rho_{ij}^{\bar{x}}\right)^2\right)z^2+{y_i}^2-2y_i\rho_{ij}^{\bar{x}}z+\left(\rho_{ij}^{\bar{x}}\right)z^2}{1-\left(\rho_{ij}^{\bar{x}}\right)^2} \\
&= \frac{z^2-2y_i\rho_{ij}^{\bar{x}}z+{y_i}^2}{1-\left(\rho_{ij}^{\bar{x}}\right)^2} \\
&= \frac{z^2-2y_i\rho_{ij}^{\bar{x}}z+{y_i}^2+{y_i}^2\left(\rho_{ij}^{\bar{x}}\right)^2-{y_i}^2\left(\rho_{ij}^{\bar{x}}\right)^2}{1-\left(\rho_{ij}^{\bar{x}}\right)^2} \\
&= \left(\frac{z-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)^2+{y_i}^2.
\end{aligned} \tag{5.112}$$

Hence, (5.111) can be written as

$$
\begin{aligned}
\frac{\partial}{\partial y_i}\Phi_2\left(y_i, y_j, \rho_{ij}^{\bar{x}}\right) &= \frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}} \int\limits_{z=-\infty}^{y_i} \frac{1}{2\pi}\exp\left(-\frac{1}{2}\left[{y_i}^2 + \left(\frac{z-\rho_{ij}^{\bar{x}} y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)^2\right]\right) dz \\
&= \varphi(y_i) \int\limits_{z=-\infty}^{y_j} \frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}} \varphi\left(\frac{z-\rho_{ij}^{\bar{x}} y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right) dz.
\end{aligned}
\tag{5.113}
$$

For solving the integral, we substitute $t := \frac{z-\rho_{ij}^{\bar{x}} y_i}{\sqrt{1-(\rho_{ij}^{\bar{x}})^2}}$, and thus $\frac{dz}{dt} = \sqrt{1-(\rho_{ij}^{\bar{x}})^2}$. This leads to

$$
\begin{aligned}
\frac{\partial}{\partial y_i}\Phi_2\left(y_i, y_j, \rho_{ij}^{\bar{x}}\right) &= \varphi(y_i) \int\limits_{t=-\infty}^{\frac{y_j-\rho_{ij}^{\bar{x}} y_i}{\sqrt{1-(\rho_{ij}^{\bar{x}})^2}}} \frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}} \varphi(t) \sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}\, dt \\
&= \varphi(y_i)\Phi\left(\frac{y_j-\rho_{ij}^{\bar{x}} y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right).
\end{aligned}
\tag{5.114}
$$

Analogously, the term (IV) of (5.109) is equivalent to

$$
\begin{aligned}
\frac{\partial}{\partial y_j}\Phi_2\left(y_i, y_j, \rho_{ij}^{\bar{x}}\right) &= \frac{\partial}{\partial y_j} \int\limits_{z=-\infty}^{y_i} (z)\Phi\left(\frac{y_j-\rho_{ij}^{\bar{x}} z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right) dz \\
&= \varphi\left(y_j\right)\Phi\left(\frac{y_i-\rho_{ij}^{\bar{x}} y_j}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right).
\end{aligned}
\tag{5.115}
$$

The derivatives (I) and (III) of (5.109) are given as

$$
\frac{dy_i(\bar{x})}{d\bar{x}} = -\frac{c_i}{\sqrt{1-{c_i}^2}} \quad \text{and} \quad \frac{dy_j(\bar{x})}{d\bar{x}} = -\frac{c_j}{\sqrt{1-{c_j}^2}}. \tag{5.116}
$$

Thus, inserting (5.114), (5.115), and (5.116) into (5.109), the derivative of the bivariate normal distribution finally results in

$$\begin{aligned}\frac{d}{d\bar{x}}\Phi_2\left(y_i(\bar{x}), y_j(\bar{x}), \rho_{ij}^{\bar{x}}\right) &= -\frac{c_i}{\sqrt{1-c_i^2}}\varphi(y_i)\Phi\left(\frac{y_j-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)\\ &\quad -\frac{c_j}{\sqrt{1-c_j^2}}\varphi\left(y_j\right)\Phi\left(\frac{y_i-\rho_{ij}^{\bar{x}}y_j}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)\\ &= \frac{dp_i(\bar{x})}{d\bar{x}}\Phi\left(\frac{\Phi^{-1}\left[p_j(\bar{x})\right]-\rho_{ij}^{\bar{x}}\Phi^{-1}\left[p_i(\bar{x})\right]}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)\\ &\quad +\frac{dp_j(\bar{x})}{d\bar{x}}\Phi\left(\frac{\Phi^{-1}\left[p_i(\bar{x})\right]-\rho_{ij}^{\bar{x}}\Phi^{-1}\left[p_j(\bar{x})\right]}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right),\end{aligned} \tag{5.117}$$

where the derivatives $\frac{dp_i(\bar{x})}{d\bar{x}}$ and $\frac{dp_j(\bar{x})}{d\bar{x}}$ are given by (5.16), which is equal to proposition (5.107).

As a next step, the derivatives of $\eta_{2,c}^{\infty}(\bar{x})$ and $\eta_{2,c}^{\mathrm{GA}}(\bar{x})$ will be calculated. With

$$\begin{aligned}\eta_{2,c}^{\infty}(\bar{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} & w_i w_j ELGD_i ELGD_j\\ &\cdot\left[\Phi_2\left(\Phi^{-1}(p_i(\bar{x})), \Phi^{-1}\left(p_j(\bar{x})\right), \rho_{ij}^{\bar{x}}\right) - p_i(\bar{x})p_j(\bar{x})\right],\end{aligned} \tag{5.118}$$

we get

$$\begin{aligned}\frac{d\eta_{2,c}^{\infty}(\bar{x})}{d\bar{x}} &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j\left[\frac{d}{d\bar{x}}\Phi_2\left(y_i(\bar{x}), y_j(\bar{x}), \rho_{ij}^{\bar{x}}\right) - \frac{d}{d\bar{x}}\left(p_i(\bar{x})p_j(\bar{x})\right)\right]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j\\ &\quad\cdot\left[\frac{d}{d\bar{x}}\Phi_2\left(y_i(\bar{x}), y_j(\bar{x}), \rho_{ij}^{\bar{x}}\right) - \left(\frac{dp_i(\bar{x})}{d\bar{x}}p_j(\bar{x}) + \frac{dp_j(\bar{x})}{d\bar{x}}p_i(\bar{x})\right)\right].\end{aligned} \tag{5.119}$$

Using the derivative of the bivariate normal distribution from (5.117) yields

$$\begin{aligned}\frac{d\eta_{2,c}^{\infty}(\overline{x})}{d\overline{x}} = & \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j \\ & \cdot \left(\frac{dp_i(\overline{x})}{d\overline{x}} \Phi\left(\frac{\Phi^{-1}\left[p_j(\overline{x})\right] - \rho_{ij}^{\overline{x}}\Phi^{-1}\left[p_i(\overline{x})\right]}{\sqrt{1-\left(\rho_{ij}^{\overline{x}}\right)^2}} \right) \right. \\ & + \frac{dp_j(\overline{x})}{d\overline{x}} \Phi\left(\frac{\Phi^{-1}\left[p_i(\overline{x})\right] - \rho_{ij}^{\overline{x}}\Phi^{-1}\left[p_j(\overline{x})\right]}{\sqrt{1-\left(\rho_{ij}^{\overline{x}}\right)^2}} \right) \\ & \left. - \frac{dp_i(\overline{x})}{d\overline{x}} p_j(\overline{x}) - \frac{dp_j(\overline{x})}{d\overline{x}} p_i(\overline{x}) \right). \end{aligned} \tag{5.120}$$

Comparing the terms on the right-hand side, it can be found that the first and second summand as well as the third and fourth summand only differ concerning the indices i and j. Due to the double sum, each combination of i and j occurs twice. Thus, (5.120) can be simplified to:[331]

$$\begin{aligned}\frac{d\eta_{2,c}^{\infty}(\overline{x})}{d\overline{x}} = & \, 2 \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j \frac{dp_i(\overline{x})}{d\overline{x}} \\ & \cdot \left(\Phi\left(\frac{\Phi^{-1}\left[p_j(\overline{x})\right] - \rho_{ij}^{\overline{x}}\Phi^{-1}\left[p_i(\overline{x})\right]}{\sqrt{1-\left(\rho_{ij}^{\overline{x}}\right)^2}} \right) - p_j(\overline{x}) \right). \end{aligned} \tag{5.121}$$

Similarly, the derivative of

$$\begin{aligned}\eta_{2,c}^{\mathrm{GA}}(\overline{x}) = & \sum_{i=1}^{n} w_i^{\,2} \left(ELGD_i^{\,2} \left[p_i(\overline{x}) - \Phi_2\left(\Phi^{-1}(p_i(\overline{x})), \Phi^{-1}(p_i(\overline{x})), \rho_{ii}^{\overline{x}}\right)\right] \right. \\ & \left. + VLGD_i p_i(\overline{x}) \right) \end{aligned} \tag{5.122}$$

[331] It has to be noticed that the conditional correlation matrix is symmetric, so we have $\rho_{ij}^{\overline{x}} = \rho_{ji}^{\overline{x}}$ for all i, j.

is given as

$$\frac{d\eta_{2,c}^{\mathrm{GA}}(\overline{x})}{d\overline{x}} = \sum_{i=1}^{n} w_i^2 \left(ELGD_i^2 \left[\frac{dp_i(\overline{x})}{d\overline{x}} - \frac{d}{d\overline{x}} \Phi_2\left(y_i(\overline{x}), y_i(\overline{x}), \rho_{ii}^{\overline{x}}\right) \right] + VLGD_i \frac{dp_i(\overline{x})}{d\overline{x}} \right). \tag{5.123}$$

Inserting the derivative of the bivariate normal distribution (5.117) finally leads to

$$\frac{d\eta_{2,c}^{\mathrm{GA}}(\overline{x})}{d\overline{x}} = \sum_{i=1}^{n} w_i^2 \frac{dp_i(\overline{x})}{d\overline{x}} \cdot \left(ELGD_i^2 \left[1 - 2\Phi\left(\frac{\Phi^{-1}[p_i(\overline{x})] - \rho_{ii}^{\overline{x}} \Phi^{-1}[p_i(\overline{x})]}{\sqrt{1 - \left(\rho_{ii}^{\overline{x}}\right)^2}} \right) \right] + VLGD_i \right). \tag{5.124}$$

5.5.5 *Moment Matching in the BET-Model*

5.5.5.1 Matching the First Moment

The expected loss of the original portfolio can be calculated as

$$\mathbb{E}\left(\tilde{L}^{\mathrm{orig}}\right) = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD \cdot \mathbb{E}\left(1_{\{\tilde{D}_{s,i}\}}\right) = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD \cdot PD_{s,i}, \tag{5.125}$$

and the expected loss of the hypothetical portfolio as

$$\begin{aligned} \mathbb{E}\left(\tilde{L}^{\mathrm{hyp}}\right) &= \sum_{i=1}^{D} \frac{1}{D} \cdot LGD \cdot \mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) = \frac{1}{D} \cdot LGD \cdot \sum_{i=1}^{D} \overline{p} \\ &= \frac{1}{D} \cdot LGD \cdot D \cdot \overline{p} = LGD \cdot \overline{p}, \end{aligned} \tag{5.126}$$

with $\mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) = \overline{p}$ for all i. Thus, matching the expectation for both portfolios leads to

$$\begin{aligned} \mathbb{E}\left(\tilde{L}^{\mathrm{orig}}\right) &\stackrel{!}{=} \mathbb{E}\left(\tilde{L}^{\mathrm{hyp}}\right) \\ \Leftrightarrow \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD \cdot PD_{s,i} &= LGD \cdot \overline{p} \\ \Leftrightarrow \overline{p} &= \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot PD_{s,i}. \end{aligned} \tag{5.127}$$

5.5.5.2 Matching the Second Moment

For the original portfolio, the variance can be calculated as

$$\begin{aligned}\mathbb{V}\left(\tilde{L}^{\text{orig}}\right) &= \mathbb{V}\left(\sum_{s=1}^{S}\sum_{i=1}^{n_s} w_{s,i}\cdot LGD\cdot 1_{\{\tilde{D}_{s,i}\}}\right)\\ &= LGD^2\cdot\mathbb{V}\left(\sum_{s=1}^{S}\sum_{i=1}^{n_s} w_{s,i}\cdot 1_{\{\tilde{D}_{s,i}\}}\right)\\ &= LGD^2\cdot\operatorname{Cov}\left(\sum_{s=1}^{S}\sum_{i=1}^{n_s} w_{s,i}\cdot 1_{\{\tilde{D}_{s,i}\}},\sum_{t=1}^{S}\sum_{j=1}^{n_t} w_{t,j}\cdot 1_{\{\tilde{D}_{t,j}\}}\right)\\ &= LGD^2\cdot\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot\operatorname{Cov}\left(1_{\{\tilde{D}_{s,i}\}},1_{\{\tilde{D}_{t,j}\}}\right)\\ &= LGD^2\cdot\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot\operatorname{Corr}\left(1_{\{\tilde{D}_{s,i}\}},1_{\{\tilde{D}_{t,j}\}}\right)\\ &\quad\cdot\sqrt{\mathbb{V}\left(1_{\{\tilde{D}_{s,i}\}}\right)}\cdot\sqrt{\mathbb{V}\left(1_{\{\tilde{D}_{t,j}\}}\right)}.\end{aligned}\tag{5.128}$$

As the default variable is Bernoulli distributed, the variance terms equal

$$\mathbb{V}\left(1_{\{\tilde{D}_{s,i}\}}\right) = PD_{s,i}\cdot\left(1-PD_{s,i}\right)\quad\text{and}\quad\mathbb{V}\left(1_{\{\tilde{D}_{t,j}\}}\right) = PD_{t,j}\cdot\left(1-PD_{t,j}\right)\tag{5.129}$$

and we obtain

$$\begin{aligned}\mathbb{V}\left(\tilde{L}^{\text{orig}}\right) &= LGD^2\cdot\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot\operatorname{Corr}\left(1_{\{\tilde{D}_{s,i}\}},1_{\{\tilde{D}_{t,j}\}}\right)\\ &\quad\cdot\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}.\end{aligned}\tag{5.130}$$

Due to the independence of the default events in the hypothetical portfolio, the variance of this portfolio is

$$\begin{aligned}\mathbb{V}\left(\tilde{L}^{\text{hyp}}\right) &= \mathbb{V}\left(\sum_{i=1}^{D}\frac{1}{D}\cdot LGD\cdot 1_{\{\tilde{D}_i\}}\right) = \frac{1}{D^2}\cdot LGD^2\cdot\mathbb{V}\left(\sum_{i=1}^{D} 1_{\{\tilde{D}_i\}}\right)\\ &= \frac{1}{D^2}\cdot LGD^2\cdot D\cdot\mathbb{V}\left(1_{\{\tilde{D}_i\}}\right) = \frac{1}{D}\cdot LGD^2\cdot\overline{p}\cdot(1-\overline{p}).\end{aligned}\tag{5.131}$$

Matching the variance terms (5.130) and (5.131) leads to

$$\mathbb{V}\left(\tilde{L}^{\text{orig}}\right) \overset{!}{=} \mathbb{V}\left(\tilde{L}^{\text{hyp}}\right)$$
$$\Leftrightarrow D = \frac{\overline{p}\cdot(1-\overline{p})}{\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \text{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)\cdot\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}. \tag{5.132}$$

5.5.6 *Interrelation of the Pairwise Default Correlation and the Asset Correlation*

Using the standard calculus for the correlation and covariance as well as the variance of a Bernoulli distributed variable, the pairwise default correlation between borrower i in sector s and borrower j in sector t can be expressed as

$$\begin{aligned}\text{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right) &= \frac{\text{Cov}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)}{\sqrt{\mathbb{V}\left(1_{\{\tilde{D}_{s,i}\}}\right)}\cdot\sqrt{\mathbb{V}\left(1_{\{\tilde{D}_{t,j}\}}\right)}} \\ &= \frac{\text{Cov}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)}{\sqrt{PD_{s,i}\cdot\left(1-PD_{s,i}\right)\cdot PD_{t,j}\cdot\left(1-PD_{t,j}\right)}} \\ &= \frac{\mathbb{E}\left(1_{\{\tilde{D}_{s,i}\}}\cdot 1_{\{\tilde{D}_{t,j}\}}\right) - \mathbb{E}\left(1_{\{\tilde{D}_{s,i}\}}\right)\cdot\mathbb{E}\left(1_{\{\tilde{D}_{t,j}\}}\right)}{\sqrt{PD_{s,i}\cdot\left(1-PD_{s,i}\right)\cdot PD_{t,j}\cdot\left(1-PD_{t,j}\right)}}.\end{aligned} \tag{5.133}$$

The expectation values of the individual default events equal $PD_{s,i}$ and $PD_{t,j}$. Similar to (5.102), assuming a normally distributed asset return, the expectation value of a simultaneous default can be written as

$$\begin{aligned}\mathbb{E}\left(1_{\{\tilde{D}_{s,i}\}}\cdot 1_{\{\tilde{D}_{t,j}\}}\right) &= \mathbb{P}\left[\left(1_{\{\tilde{D}_{s,i}\}}=1\right)\wedge\left(1_{\{\tilde{D}_{t,j}\}}=1\right)\right] \\ &= \mathbb{P}\left(\tilde{a}_{s,i}\le\Phi^{-1}(PD_{s,i}), \tilde{a}_{t,j}\le\Phi^{-1}(PD_{t,j})\right) \\ &= \Phi_2\left(\Phi^{-1}\left(PD_{s,i}\right), \Phi^{-1}\left(PD_{t,j}\right), \text{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right)\right).\end{aligned} \tag{5.134}$$

Thus, we get

$$\text{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right) = \frac{\Phi_2\left(\Phi^{-1}\left(PD_{s,i}\right), \Phi^{-1}\left(PD_{t,j}\right), \text{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right)\right) - PD_{s,i}\cdot PD_{t,j}}{\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}. \tag{5.135}$$

5.5.7 Expected Number of Defaults in the Infectious Defaults Model

Due to the homogeneity of the portfolio and the stochastic independence of all indicator variables, the expected number of defaults is

$$
\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}}\right) &= n \cdot \mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) \\
&= n \cdot \mathbb{E}\left(\tilde{Z}_i\right) \\
&= n \cdot \mathbb{E}\left(\tilde{X}_i + \left(1 - \tilde{X}_i\right) \cdot \left[1 - \prod_{j \neq i} \left(1 - \tilde{X}_j \cdot \tilde{Y}_{j,i}\right)\right]\right) \\
&= n \cdot \mathbb{E}\left(\tilde{X}_i + \left(1 - \tilde{X}_i\right) \cdot \left[1 - \left(1 - \tilde{X}_j \cdot \tilde{Y}_{j,i}\right)^{n-1}\right]\right) \\
&= n \cdot \left(\mathbb{E}\left(\tilde{X}_i\right) + \left(1 - \mathbb{E}\left(\tilde{X}_i\right)\right) \cdot \left[1 - \left(1 - \mathbb{E}\left(\tilde{X}_j\right) \cdot \mathbb{E}\left(\tilde{Y}_{j,i}\right)\right)^{n-1}\right]\right) \\
&= n \cdot \left(p + (1 - p) \cdot \left[1 - (1 - p \cdot q)^{n-1}\right]\right) \\
&= n \cdot \left(1 - (1 - p) \cdot (1 - p \cdot q)^{n-1}\right).
\end{aligned}
\tag{5.136}
$$